DOCUMENT RESUME

ED 160 415 SE 025 038

AUTHOR	Haag, V. H.; And Others
TITLE	Introduction to Algebra, Teacher's Commentary, Part I, Unit 45. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	65
NOTE	342p.; For related documents, see SE 025 036-039; Contains occasional light and broken type
EDRS PRICE	MF-$0.83 HC-$18.07 Plus Postage.
DESCRIPTORS	*Algebra; Curriculum; *Grade 9; *Instruction; Mathematics Education; Number Concepts; Secondary Education; *Secondary School Mathematics; Set Theory; *Teaching Guides
IDENTIFIERS	Mathematical Sentences; *School Mathematics Study Group

ABSTRACT

This is part one of a two-part manual for teachers using SMSG text materials for grade 9 students whose mathematical talents are underdeveloped. The overall purpose for each of the chapters is described and the mathematical development detailed. Background information for key concepts, answers for all exercises in each chapter, and suggested test items are provided. Chapter topics include: (1) sets and the number line; (2) numerals, sentences, and variables; (3) open sentences and truth sets; (4) properties of operations; (5) open sentences and word sentences; (6) real numbers and the four basic operations; and (7) properties of order. (MN)

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SCHOOL MATHEMATICS STUDY GROUP

Introduction to Algebra

Teacher's Commentary, Part I

REVISED EDITION

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Distributed for the School Mathematics Study Group
by A. C. Vroman, Inc., 367 Pasadena Avenue, Pasadena, California

Financial support for School Mathematics
Study Group has been provided by the
National Science Foundation.

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PREFACE TO TEACHERS

This text has been written for the ninth grade student whose mathematical talent is underdeveloped. The subject matter presented is essentially that which appears in the School Mathematics Study Group text: First Course in Algebra. This is part of the body of mathematics which members of the Study Group believe is important for all well educated citizens in our society. It is also the mathematics which is important for the pre-college student as he prepares for advanced work in the field of mathematics and related subjects.

It is the hope of the panel that this material will serve to awaken the interest of a large group of students who have mathematical ability which has not yet been recognized. It is hoped also that this text will contribute to the understanding of fundamental concepts for those students whose progress in mathematics has been blocked or hampered through rote learning or inappropriate curriculum. However this text is not offered as appropriate content for the slow learners among the non college-bound students.

The mathematics which appears in the text is not of the type normally called "business" or "vocational" mathematics; nor is it intended that this serve as a terminal course. Rather, as the title clearly states, this is an introduction to Algebra which will provide the student with many of the basic concepts necessary for further study.

Some of the important features of the text are the following:

(1) The reading level is appropriate for the kind of students for whom the text is written.

(2) In order to achieve the objective of introducing one new concept at a time sections are divided into subsections, each including exercises.

(3) New concepts are introduced through concrete examples.

(4) Easy drill material is included in the exercises.
(5) Chapter summaries and adequate sets of review problems are provided.
(6) Terminology is kept to a minimum.
(7) A glossary of important terms and definitions is included at the end of each of the four parts.

Some general suggestions for the use of the text are offered below.

Reading.

As is the case with all SMSG texts this text was written with the expectation that it can and will be read by the student. Since many students are not accustomed to reading a book on mathematics, it will be necessary to assist them in learning to make the best use of the book.

Teachers report that at the beginning of the course they find it best to read the text aloud while students read silently. When students eventually do the reading on their own they need to be reminded again and again of the necessity for rereading some of the sentences. It is hoped that by the end of the year they will have gained a good measure of competence in reading mathematics.

Check Your Reading.

The text provides sets of questions titled Check Your Reading which are concerned with the ideas in the material which the student has just read.

It would be wise to start a class period by reviewing the reading questions from the preceding day or the preceding two days. The student who was not able to discuss a question when it was first encountered would have the opportunity to do so in the review.

Problem Sets.

The text has an ample supply of exercises. They are graded in each list so that the most difficult are at the end of the list. In an exercise which has parts the teacher should use as many of them as seems best for the particular class situation.

Problems have been included which may be omitted without any loss of continuity. Among them are starred problems which are more difficult than others. Problems of this type as well as the challenge problems which appear at the end of each of the four parts, might well be appropriate for the "extra credit" part of the assignments.

This text is in four parts. In the directions sent out to "tryout" centers during the past two years teachers were advised to use their own judgement as to how rapidly they should introduce the material to their students. The reports of the teachers indicate that it takes more than one year of study for students of average ability to complete the four parts successfully. It is not clear as yet how well students of lower than average ability can learn algebra from this text.

A comparison experiment conducted recently by the Minnesota National Laboratory showed that college capable students studying from this text performed as well on SMSG unit tests as students of like ability studying from the text First Course in Algebra.

Chapter 1

SETS AND THE NUMBER LINE

In this chapter we use the non-negative rational numbers and the basic operations upon them as a familiar background for the introduction of concepts and procedures which may be new to the pupil. We consider briefly two of the indispensible tools for our study of the structure of the real number system -- sets and the number line.

One of the great unifying and simplifying concepts of all mathematics, the idea of set, is of importance throughout the course in many ways: in classifying the numbers with which we work, in examining the properties of the operations upon these numbers, in solving equations and inequalities, in factoring polynomials, in the study of functions, etc.

Since most students have not studied about sets before entering this course, and since the basic notions of set are usually grasped quite readily, it seems a good topic, from a motivational standpoint, with which to start the course. We move on quickly from this first discussion of sets, however, postponing much work with operations on elements of sets and with closure, so as to get quickly to the presentation of variable (in Chapter 2). This is done largely because (1) teachers and students expect the early introduction of variable and (2) our study of the structure of the number system can begin with the idea of variable.

Next we place the number line before the student. Here again is a concept that is of use throughout the course. It is the device for picturing many of the ideas about numbers and operations on them. This is immediately apparent as the graphing of sets is introduced and is followed in the final section of the chapter by addition and multiplication on the number line.

Pupils who have studied SMSG Mathematics for Junior High School will have had a little experience with sets and the number line. They may be able to go through parts of this chapter a little more quickly than other students, but the treatment is sufficiently different that nothing should be omitted.

The teacher is referred to Haag, Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 2, Section 1.

1-1. Sets.

Though the first sets listed at the outset of the chapter are not examples of sets of numbers, we move quickly in the text to consideration of such sets. Though non-numerical sets may be of interest, a prolonged discussion of them would constitute a diversion from the basic purpose of the course.

The concept of set is introduced by making use of the student's experience. You may find it necessary or desirable to give several other examples.

We do not introduce much of the standard set notation such as set builder notation, $\in$, $\supset$, $\subset$, $\cup$, $\cap$, because the topics to which these notations are particularly well adapted are probably too widely separated in the book for retention. There is, however, no objection to the teacher using any of these if he so desires. Certainly, if the class already has a background including set notation, the teacher should make use of it.

Braces are introduced as a means of recognizing sets and as a means of listing sets.

Study Guide; page 2:

1. Stress the idea that "set" will be used throughout the course.

Answers to Problem Set 1-1a; pages 2-3:

1. (a) {9, 19, 29, 39, 49}
   (b) {3, 13, 23, 33, 43}
   (c) {10, 20, 30, 40}
2. The set in (c). It has 4 elements.
3. (a) {d, e, f, g, h, i, j}
   (b) {o, e, i, a}
   (c) {i, s, p} (Point out that the letter is listed only once even though it occurs more than once.)
4. (a) $\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$

(b) $\{\frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}\}$

(c) $\{\frac{5}{2}, \frac{7}{3}, \frac{8}{3}, \frac{11}{5}, \frac{12}{5}, \frac{13}{5}, \frac{14}{5}\}$

Problems preceded by the asterisk * are more challenging than others in the same set of exercises. Such problems are included primarily for the brighter and more curious student, and the use of these problems with an entire class may consume time needed later in the year to complete the basic work of the course. The teacher will have to decide as he reaches each such problem whether time and the ability of his students permit him to deal with the problem with the class as a whole.

*5. (a) {California, Connecticut, Colorado}
(b) {N.Y., N.J., N.H., N.C., N.D., N.M., Neb., Nev.}
(c) {Hawaii}
(d) There are no elements in this set.

Pages 3-4. It should be pointed out that there are two methods of describing sets. A set can be listed with the elements enclosed in braces, or a set can be described with a verbal description. It is important to note that in some cases a verbal description and a listing are equally adequate in describing a set. However, there are sets which can be described only in one of the two ways. On one hand, for example, is the set {2, 3, 5, 7, 8}, which is not easily described in words; on the other hand, there is the null set, which cannot be listed and must be described in some other manner.

Pages 4-5. We introduce the technique for listing sets which have many elements and sets that are infinite. We use the common notation of the three dots ". . ." which mean "and so forth" or "continuing in the same pattern". Depending upon the class, this notation may or may not need more explanation.

The representation of a set by a capital letter is introduced. The student should understand that this is simply a way of naming the set. We then define the set of counting numbers, whole numbers, multiples of 3 (to clarify the concept of a multiple), even numbers, and the set of odd numbers.

Answers to Problem Set 1-1b; pages 6-7:

1. (a) {1, 2, 3, . . ., 12}
   (b) {0, 1, 2, . . ., 10}
   (c) {11, 12, 13, . . .}
   (d) {0, 7, 14, . . ., 49}
   (e) {0, 3, 6, . . ., 27}
   (f) {0, 2, 4, . . ., 12}
   (g) {16, 18, 20, . . .}
   (h) {11, 13, 15, . . .}
   (i) {1, 3, 5, . . ., 39}
   (j) $\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots, \frac{6}{7}\}$

2. (a) A is the set of all even numbers less than 7.
   (b) B is the set of all odd numbers.
   (c) C is the set of all multiples of 6.
   (d) D is the set of all whole numbers.
   (e) E is the set of all multiples of 4.
   (f) F is the set of all whole (or counting) numbers greater than 11.
   (g) G is the set of all odd numbers greater than 21 (or 22).
   (h) H is the set of all odd numbers less than 19 (or 18).
   (i) I is the set of all multiples of 6 which are less than 72 (or 67).
   (j) J is the set of all even numbers greater than 26 and less than 100.
   (k) K is the set of all 30-day months. (Other verbal descriptions are possible).

3. The set obtained by dividing each element of the set of even numbers by two is the set: {0, 1, 2, . . .}, which is the set of whole numbers.

Many of the problem sets in this chapter are short, and the teacher may wish to cover more than one problem set in a day. For most students, the short problem sets should suffice to convey the idea of sets which are needed in this course. The teacher is cautioned not to dwell on these sections at length, nor to prolong greatly the exercise work on sets, for it is the algebraic structure of the real number system, rather than the study of sets for their own sake, that constitutes the heart of the course.

Answers to Problem Set 1-1c; pages 8-10:

1. B is a subset of A
   D is a subset of A
   E is a subset of A
   H is a subset of A

   C is not a subset of A because 30 is an element of C and not an element of A.

   F is not a subset of A because 0 is an element of F and not an element of A.

   G is not a subset of A because 27, 29, . . . are elements of G and not elements of A.

2. T = {1, 4, 9, 16}
   R = {1, 4}
   (a) No, R does not contain 2 as an element.
   (b) Yes
   (c) Yes
   (d) No. 9, 16 are elements of T but not of S.

3. K = {1, 2, 3, 4, 9, 16}
   (a) S, T, R, K are all subsets of K.
   (b) R is a subset of R.
   (c) K has the most elements.
   (d) R has the fewest elements.

4. (a) We have defined an odd number as "one more than an even number." Hence, if we add two odd numbers we will have 2 more than an even number--which will be an even number.

(b) From our definition of an odd number, multiplying an odd number by an odd number would always result in an "extra" one, so the product is always odd.

5. If T = {1, 2, 3, 4}, the set of sums of pairs of T is S = {2, 3, 4, 5, 6, 7, 8}. S is not a subset of T because the elements 5, 6, 7, 8 are not in T.

6. If Q = {0, 1}, then the set of products of pairs is P = {0, 1}. P is a subset of Q.

7. If R = {0, 1, 2}, the set of sums of pairs of R is S = {0, 1, 2, 3, 4} and the set P of products of pairs of R is P = {0, 1, 2, 4}; neither P nor S is a subset of R.

*8. The subject of closure introduced in this problem will be dwelt upon thoroughly later so it should probably be left as an interest problem at this time and should not be allowed to distract the class from more immediate ideas.

(a) T is not closed under addition.

(b) Q is closed under multiplication but not under addition.

(c) R is not closed under either multiplication or addition.

(d) N is closed under both multiplication and addition.

Comment on problems 2 and 3.

Experience shows that students usually have difficulty understanding the directions given for these two problems, regardless of the care with which the instructions are written. Here we are touching for the first time the ideas of "intersection" and "union" of two sets. These will be hit again in various contexts; thus, it is not necessary for the teacher to make an all-out production of problems 2 and 3. The difficulties here can be eased by means of a dialogue between teachers and class in which it is made clear that

1) the elements in R and in S consist of those elements common to R, S;

2) the elements in R or in S consist of those elements either in R or in S or both.

After the class succeeds in understanding these two operations on sets, be sure that the words and and or remain the key words rather than the words "both", "common", "either", etc. There is a good reason for this, because very soon in the course (Chapter 3) we will meet conjunctions and disjunctions of sentences in which the intersections and unions of sets will be implied by and and or, respectively.

Pages 10-11. The teacher should be aware of three common errors made by students in working with the empty set. The most common error is the confusion of $\{0\}$ and $\emptyset$, and this is warned against in the text, but may need further emphasis by the teacher. A less significant mistake is to use the words "an empty set" or "a null set" instead of "the empty set" or "the null set". There is but one empty set though it has many descriptions. A third error is the use of the symbol, $\{\emptyset\}$, instead of just $\emptyset$.

The statement that the null set is a subset of every set may cause some difficulty. The teacher should point out that to say that every element of A is an element of B means that there is no element in A which is not in B. The null set $\emptyset$ is a subset of the set $\{1, 2, 3\}$ since $\emptyset$ has no elements which are not in the set $\{1, 2, 3\}$.

Answers to Problem Set 1-1d; pages 11-12:

1. (a) $A = \{2\}$; therefore it is not $\emptyset$.
   (b) $B = \emptyset$
   (c) $C = \emptyset$
   (d) This is not the null set but the set $\{0\}$
   (e) $\emptyset$
   (f) $\{0\}$, not $\emptyset$

2. The list of subsets of B is ∅
{1}
{2}
{3} There are eight subsets.
{1, 2} $2 \times 2 \times 2 = 2^3$
{1, 3}
{2, 3}
{1, 2, 3}

3. The list of subsets of C is
∅, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}.

   There are 16 subsets. $2 \times 2 \times 2 \times 2 = 2^4$

4. For a set consisting of n elements, the number of subsets is 2^n. This problem is included to help discover the student who has the ability to generalize. Do not consider this as something for the entire class to master at this time, certainly not the notation 2^n.

Pages 12-14. The number line is used as an illustrative and motivational device, and our discussion of it is quite intuitive and informal. As was the case with the preceding section, more questions are raised than can be answered immediately.

Present on the number line implicitly are points corresponding to the negative numbers, as is suggested by the presence in the illustration of the left side of the number line. Since, however, the plan of the course is to move directly to the consideration of the properties of the operations on the non-negative numbers, anything more than casual recognition of the existence of the negative numbers at this time would be a distraction to the student.

The idea of successor is important. Suppose you begin with the counting number one. The successor is "one more", or 1 + 1. The successor of 105 is 105 + 1, or 106; of 100,000,005 is 100,000,006. This implies that whenever you think of a whole number, however large, it always has a successor. To the pupil should come the realization that *there is no last number*. An interesting reference for the student is Tobias Dantzig, *Number, the Language of Science*, pp. 61-64.

The use of the term "infinitely many" on the part of the student and teacher should help the student *avoid* the noun "infinity," and with it the temptation to call "infinity" a numeral for a large number.

The emphasis here is on the fact that a *coordinate* is the *number* which is *associated* with a point on the line. "Coordinate", "associated", and "corresponding to" must eventually become part of the pupil's vocabulary. He must not confuse *coordinate* with point, nor *coordinate* with the name of the number.

The distinction between *number* and *name of a number* comes up here for the first time. Do not make an issue of it at this time, for it is dealt with explicitly at the beginning of Chapter 2.

Answers to Problem Set 1-2a; page 15:

1. $S = \{1, 6, 11, 16, 21, \ldots, 46\}$ (a list description)
2. (a) finite
   (b) infinite
   (c) infinite
   (d) finite
   (e) finite
   (f) infinite

Pages 15-17. Here we picture the number line, the points being labeled with rational numbers. You may want to point this out to the students after they have read at the top of page 17. We must be careful to observe that the general statement on page 17 concerning rational numbers is not a definition, since it does not take into account the negative numbers. Do not make an issue of this with the students; for the moment we merely want them to have the idea that these numbers are among the rationals.

It is also possible to say that a number represented by a fraction indicating the division of a whole number by a counting number is a rational number. This statement may be of interest since it is expressed in terms of these recently defined sets, but the statement in the text has the advantage that the exclusion of division by zero is explicit. "$\frac{34}{10}$", "$\frac{14}{2}$", "$\frac{8}{3}$", "$\frac{9}{2}$" are some possible names for these numbers.

A rational number may be represented by a fraction, but some rational numbers may also be represented by other numerals, such as 1.333... and 1.42. The number line illustration on page 16 gives the name "2" as well as the fractions "$\frac{4}{2}$", "$\frac{6}{3}$", "$\frac{8}{4}$" to name the number 2.

The same diagram makes clear that not all rational numbers are whole numbers. The students may have seen some fractions that do not represent rational numbers, such as, $\frac{\sqrt{2}}{2}$, $\frac{4\pi}{3}$, etc. They will have to be reminded that so-called "decimal fractions" are not by this definition fractions.

It is necessary to keep the words "rational number" and "fraction" carefully distinguished. Later on in the course, it will be seen that the meaning of the term "fraction" includes any expression, also involving variables, which is in the form of an indicated quotient.

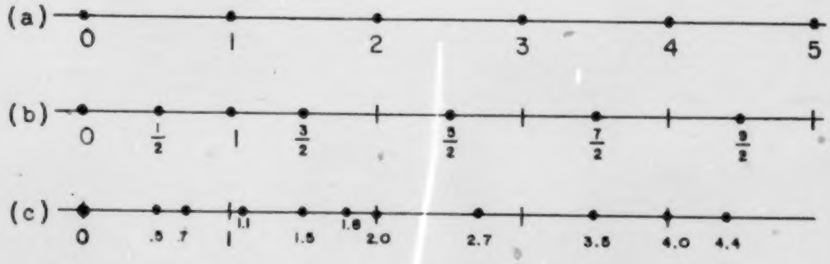
Pages 17-19. The idea of "density" of numbers is being initiated here. By density of numbers we mean that between any two numbers there is always another, and hence that between any two numbers there are infinitely many numbers. This suggests that on the number line, between any two points there is always another point, and, in fact, infinitely many points. We refer here to "points" in the mathematical, rather than physical, sense-- that is, points of no dimension. Because the student may not be thinking of points in this way he may not intuitively feel that between any two points on the number line other points may be located. Therefore, he is shown "betweenness" for numbers first; then, taking these numbers as coordinates, he can infer "betweenness" of points on the number line.

The fact that there are points on the number line which do not correspond to rational numbers should arouse the students' curiosity. Do not expand on this at this time, however. Irrational numbers will be introduced at a later time, as coordinates of such points.

At this point in the course, it is hoped that the student will accept the fact that every point to the right of 0 on the number line can be assigned a number. He may not accept the fact that not every such point has a rational number as its coordinate, but this fact need not be emphasized until Chapter 12. He may also be impatient to assign numbers to points to the left of 0. For the time being, until Chapter 6, we shall concentrate on the non-negative real numbers. This set of numbers, including 0 and all numbers which are coordinates of points to the right of 0, we call the set of numbers of arithmetic. After we establish the properties of operations on these numbers (in Chapters 2 and 4) we shall consider the set of all real numbers which includes the negative numbers (in Chapter 6). Then in Chapters 7, 8, 9, and 10, we spell out the properties of operations on all real numbers.

Answers to Problem Set 1-2b; pages 20-21:

1.

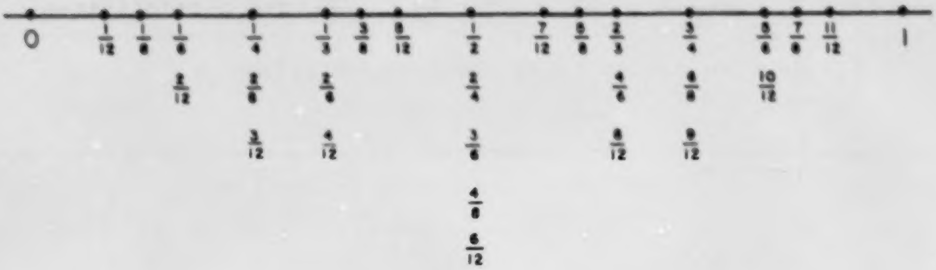


2. The student should circle the points labeled 0, 1, 2, 3, 4, 5, in (a)

   0, 1 in (b)

   0, 1, 2, 4, in (c)

3. This problem represents a very good ruler exercise. If time is a factor you may choose to omit it.



(a) largest, $\frac{11}{12}$; larger $\frac{23}{24}$, $\frac{47}{48}$, etc.

smallest, $\frac{1}{12}$; smaller, $\frac{1}{13}$, $\frac{1}{14}$, $\frac{2}{48}$ etc.

The student may notice the sequence of $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{12}$ and give the answer $\frac{12}{13}$, $\frac{13}{14}$, etc. If he doesn't, point this out.

(b) $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$

(c) $\frac{1}{8}$ is a possible answer.

$\frac{1}{12} = \frac{2}{24}$, $\frac{2}{12} = \frac{4}{24}$; between $\frac{2}{24}$ and $\frac{4}{24}$ is $\frac{3}{24}$ or $\frac{1}{8}$.

Of course others are possible such as $\frac{5}{48}$, $\frac{7}{48}$.

(d) Infinitely many.

4. (a) Infinitely many. Infinitely many.

(b) $\frac{5}{2}$, $\frac{7}{3}$ are possibilities; $\frac{5}{1000}$, $\frac{51}{10,000}$ possibilities

(c) There is none--no matter what one is offered as "next", another can be found between this number and 2. This should provoke some interesting discussion!

5. $4, \frac{4}{1}, \frac{8}{2}$, 7 - 3, 3 + 1, 2 × 2 are possibilities.

6. $\frac{6}{8}, \frac{15-3}{16}, \frac{1\frac{1}{2}}{2}$, .75, $\frac{27}{36}, \frac{300}{400}$ are possibilities.

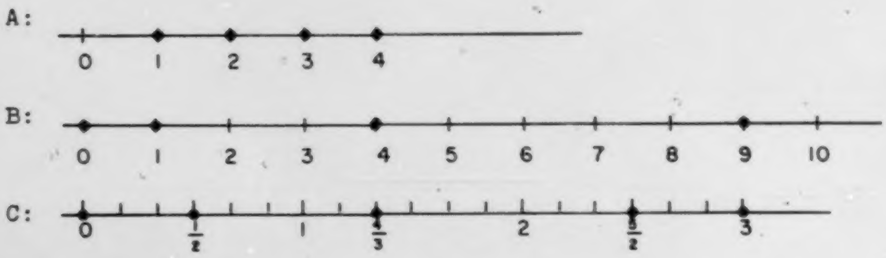
7. Here we are building the idea of order:
The point with coordinate 3.5 is to the right of the point with coordinate 2. 3.5 is greater than 2.
The point with coordinate 1.5 is to the left of the point with coordinate 2. 1.5 is less than 2.

8. (a) finite $\{\frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, \frac{7}{6}, \frac{11}{6}, \frac{8}{7}, \frac{9}{7}, \frac{10}{7}, \frac{11}{7}, \frac{12}{7}, \frac{13}{7}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8}, \frac{10}{9}, \frac{11}{9}, \frac{13}{9}, \frac{14}{9}, \frac{16}{9}, \frac{17}{9}\}$
omitting those which name the same number.
(b) infinite
(c) infinite

Pages 21-22. It should be pointed out that the graph of a set is simply the points marked on the number line.

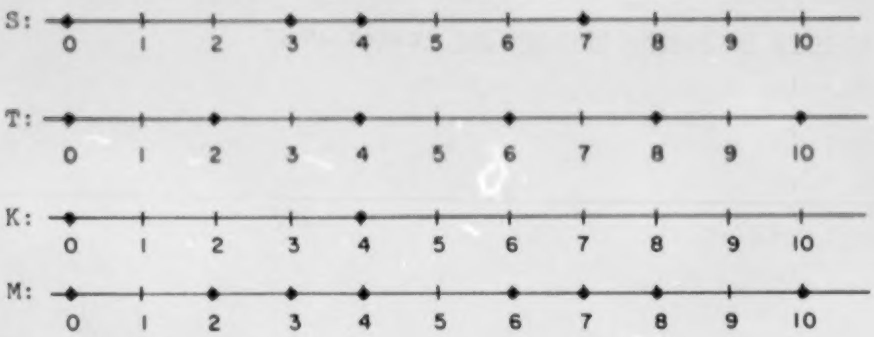
Answers to Problem Set 1-2c; pages 22-23:

1.



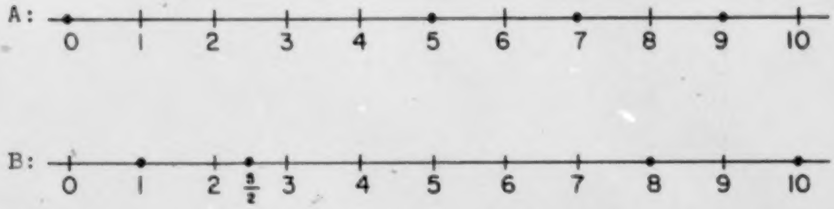
2. (a) If S = {0, 3, 4, 7} and T = {0, 2, 4, 6, 8, 10}, then K = {0, 4} and M = {0, 2, 3, 4, 6, 7, 8, 10}

(b)


S:
0 1 2 3 4 5 6 7 8 9 10
T:
0 1 2 3 4 5 6 7 8 9 10
K:
0 1 2 3 4 5 6 7 8 9 10
M:
0 1 2 3 4 5 6 7 8 9 10

(c) The points on the graph of K are projections of the points appearing simultaneously on S and T. The points on the graph of M are the projection of every point on S and every point on T. The student need not, of course, answer in these terms.

3. (a)


A:
0 1 2 3 4 5 6 7 8 9 10
B:
0 1 2 $\frac{5}{2}$ 3 4 5 6 7 8 9 10

(b) If C is the set of numbers which are elements of both A and B (meaning in A and in B), then C has no elements.

(c) C is the empty set (or the null set).

Pages 23-24. This list should not be used as a teaching aid but as a guide for the student.

Review Problem Set: pages 24-26:

The review problems can be used in a variety of ways. They may be used for homework. They may be used for test items. Problem *12 should not be given to every student. It involves a very subtle idea involving infinite subsets.

Answers to Review Problem Set; pages 24-27:

1. {0, 3, 6, 9, 12, . . ., 48}
2. {0, 3, 6, . . ., 48}
3. {0, 6, 12, . . ., 48}
4. The set of multiples of 3 is not a subset of the set of multiples of 6 because there are elements in the first set not appearing in the second. For example, the number 9 is a multiple of 3 but it is not a multiple of 6. The set of multiples of 6 *is* a subset of the set of multiples of 3.
5. The set of all even numbers greater than 8. Other descriptions are possible.
6. "The set of all odd numbers from 7 to 59 inclusive" is one of the possible descriptions.
7. The empty set.
8. (a) 18 elements
   (b) 25 elements
   (c) infinitely many elements
   (d) 34 elements (don't forget zero)
   (e) 101 **elements**
   (f) infinitely many elements
   (g) infinitely many elements
9. If S = {5, 7, 9} and T = {0, 2, 4, 6, 8, 10}
   (a) then K = $\emptyset$
   K is a subset of S and of T. All three are finite sets.
   (b) M = {0, 2, 4, 5, 6, 7, 8, 9, 10}
   M is not a subset of S. T is a subset of M. M is a finite set.
   (c) R = {5, 7, 9}
   R is a subset of M, of S.
   (d) A = $\emptyset$. A has no elements. A is the empty set.
   (e) A and K are the same.

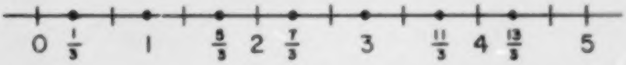
(f) Subsets of finite sets are always finite.

(g) The set D of all rational numbers from 0 to 10 inclusive is not a finite set. This illustrates the interesting idea that it is not sufficient to be able to name the last number to be able to count the set.

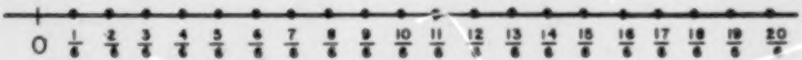
S is a subset of D.
Every infinite set does have finite subsets.
D is a subset of D.

Infinite sets can have infinite subsets, for example, the set of counting numbers is a subset of the set of rational numbers, or of the whole numbers.

10. (a)



(b)



$C = \{\frac{1}{3}, \frac{3}{3}, \frac{5}{3}, \frac{7}{3}, \frac{9}{3}\}$

$\frac{3}{3} = 1$ and $\frac{9}{3} = 3$ are whole numbers.
1 and 3 are counting numbers.
All the elements of the set are rational numbers.

11. $\frac{22}{7}$ is between the whole numbers 3 and 4.

$\frac{22}{7}$ is greater than 3.1.

The point with coordinate $\frac{22}{7}$ lies to the right of 3.1.

$\frac{22}{7}$ lies between 3.1 and 3.2.

page 26

12. The teacher should not feel compelled to use class time for this problem since the ideas may be lost on the class. However, it can lead to an interesting discussion if enough of the class will benefit from it. The obviously capable individual in the class should have the opportunity to do it. This is a much more useful definition of an infinite set than has been developed in the text.

The set of multiples of 3 is a proper subset of the set of whole numbers since it does not include the elements 1, 2, 4, 5, . . . as a partial list. One possible one-to-one correspondence between the set of whole numbers and multiples of 3 is

$$\begin{array}{ccccc} 0 & 1 & 2 & & n \\ \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow \\ 0 & 3 & 6 & & 3\times n \end{array}$$

where n represents any whole number.

For the superior student it could be pointed out that mathematicians take this as a definition of an infinite set: A set is infinite if it can be placed in one-to-one correspondence with a proper subset of itself..

Suggested Test Items

(The "suggested test items" which follow are not intended to comprise a balanced or complete test, but are, as the title implies, questions which seem suitable for inclusion in a test on this chapter.)

1. Are the following sets finite or infinite? If it is possible, list the elements of each.
   - (a) The people in this classroom today.
   - (b) All multiples of 3.
   - (c) All counting numbers less than 7.

(d) all whole numbers which are not multiples of 5.

(e) all numbers between 0 and $\frac{1}{4}$.

2. (a) Given set S = {0, 1, 2, 3, 4}. Find set T, the set of products of each element of set S and 1. Is T a subset of S?

   (b) Given set A = {0, 2, 4, 6, 8}. Find set B, the set of products of each element of set A and 0. Is B a subset of A? Is B the empty set?

3. Describe in words each of the following sets
   (a) {1, 3, 5, 7, . . .}
   (b) {0, 5, 10, 15, . . .}
   (c) {0, 1, 2, 3, 4}
   (d) $\emptyset$

4. Given set N = {1, 2, 3, 4, 8, 9, 12, 16}.
   (a) Find the subset R consisting of all elements of N which are squares of whole numbers.
   (b) Find set K of the odd numbers in set N.
   (c) Find set A of the squares of the elements of N.
   (d) Find set B whose elements are each 3 more than twice the corresponding element of N.
   (e) Find set C, the set of all numbers which are elements of both N and B.
   (f) Find set D, the set of all numbers which are elements of either N or B or both.

5. Consider each of the following sets, and for those which are finite list the elements, if possible. If the set is the empty set, write the usual symbol, $\emptyset$.
   (a) All counting numbers less than 1.
   (b) All whole numbers less than 1.
   (c) All numbers less than 1.
   (d) All counting numbers such that 10 times the number is greater than the number itself.
   (e) All whole numbers such that 10 times the number is equal to the number times itself.

6. (a) Draw a number line and locate the points whose coordinates are:

$\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{9}{3}, 0, .6, 1.2, 1.5, 1.8, 3.5, 4$

(b) Which of these coordinates are counting numbers? Whole numbers? Rational numbers?

(c) On the number line how is the point with coordinate 5.4 located with respect to the point with coordinate 4? with coordinate 6.2?

7. (a) Is $\frac{3}{7}$ to the left of $\frac{15}{21}$ on the number line?

(b) Show the graph of the set $K = \{0, 3, 7\}$.

(c) Write 3 other names that could be used for the coordinate 3.

8. If A is the set of all whole numbers less than 20 which are not multiples of either 2, 3, or 5,

(a) list the elements of set A;

(b) draw the graph of set A.

9. List two numbers between $\frac{1}{8}$ and $\frac{2}{8}$. How do you know that they are between $\frac{1}{8}$ and $\frac{2}{8}$?

*10. State S, the set of all whole numbers.

(a) Is it finite or infinite?

(b) Is it closed under addition? Explain why.

(c) Is it closed under multiplication? Explain why.

(d) Is it closed under the operation of finding the average of two numbers? Show why.

State T, the set of all odd numbers, and answer questions (a) through (d).

State R, the set of all odd numbers less than 8 and answer questions (a) through (d).

Answers to Suggested Test Items

1. (a) finite {Ann, Mary, Peter, . . ., John} Really depends on the class.
   (b) infinite
   (c) finite {1, 2, 3, 4, 5, 6}
   (d) infinite
   (e) infinite

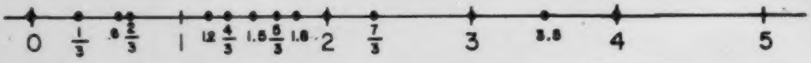
2. (a) T = {0, 1, 2, 3, 4} Yes, T is a subset of S.
   (b) B = {0}. Yes, B is a subset of A. No, B is not the empty set.

3. (a) The set of odd numbers.
   (b) The set of multiples of 5.
   (c) The set of whole numbers less than 5, or the set of whole numbers from 0 to 4, inclusive.
   (d) The empty set.

4. (a) R = {1, 4, 9, 16}
   (b) K = {1, 3, 9}
   (c) A = {1, 4, 9, 16, 64, 81, 144, 256}
   (d) B = {5, 7, 9, 11, 19, 21, 27, 35}
   (e) C = {9}
   (f) D = {1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 16, 19, 21, 27, 35}

5. (a) Ø
   (b) {0}
   (c) infinite set
   (d) infinite set
   (e) {0}

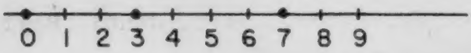
6. (a)



   (b) 2, 3, 4 are counting numbers.
   0, 2, 3, 4 are whole numbers.
   All are rational numbers.
   (c) This is to the right of the point whose coordinate is 4. It is to the left of the point whose coordinate is 6.2.

7. (a) Yes, $\frac{3}{7}$ is to the left of $\frac{15}{21}$ on the number line.

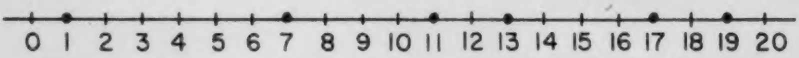
(b)



(c) Among several possibilities are $\frac{6}{2}$, $\frac{3}{1}$, 2 + 1, 5 - 2, etc.

8. (a) A = {1, 7, 11, 13, 17, 19}

(b)



9. Among several possibilities are $\frac{3}{16}$, $\frac{4}{24}$, $\frac{5}{24}$, $\frac{5}{32}$, $\frac{6}{32}$, etc.

$\frac{1}{8} = \frac{2}{16}$ and $\frac{2}{8} = \frac{4}{16}$, thus $\frac{3}{16}$ is between $\frac{1}{8}$ and $\frac{2}{8}$.

Or, $\frac{1}{8} + \frac{2}{8} = \frac{3}{8}$. And $\frac{3}{8} \div 2 = \frac{3}{16}$.

10. S = {0, 1, 2, 3, . . .}
   (a) infinite
   (b) Yes, any element in set S added to any element in set S produces an element in the set S.
   (c) Yes, same as above.
   (d) No. $\frac{7 + 8}{2} = \frac{15}{2}$ and $\frac{15}{2}$ is not an element of the set of whole numbers.

T = {1, 3, 5, 7, . . .}
   (a) infinite
   (b) No, since 1 + 3 = 4 and 4 is not an element of the set T.
   (c) Yes, same as (b) above.
   (d) No, since $\frac{1 + 3}{2} = 2$ and 2 is not an element of the set T.

R = {1, 3, 5, 7}
   (a) finite
   (b) No. 1 + 3 = 4 and 4 is not an element of R.
   (c) No. 3 × 5 = 15 and 15 is not an element of R.
   (d) No. $\frac{1 + 3}{2} = 2$ and 2 is not an element of R.

Chapter 2

NUMERALS, SENTENCES, AND VARIABLES

For background in the topics included in this chapter the teacher is referred to Haag, Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 3, Sections 1 and 2, and Chapter 6, Section 1.

2-1. Numbers and Their Names.

The aim of this section is to bring out the distinction between numbers themselves and the names for them and also to introduce the notion of a phrase. Along the way a number of important conventions used in algebra are pointed out.

We do not want to make a precise definition of "common name." The term is a relative one and should be used quite informally. Note that some numbers do not have what we would wish at this time to call a common name, such as π $\sqrt{2}$, while some may have several common names (e. g. $\frac{1}{2}$, 0.5 and $2\frac{1}{2}$, $\frac{5}{2}$, etc.).

The ideas of indicated sum and indicated product are very handy, particularly in discussing the distributive property, and will be used frequently. They are also useful to counteract the tendency, encouraged in arithmetic, to regard an expression such as "4 + 2" not as the name of a number but rather as a command to add 4 and 2 to obtain the number 6. This point of view makes it difficult for a pupil to accept such expressions as names of anything. In passing, you may wish to mention to the class indicated quotients and indicated differences. Some may already be familiar with "indicated quotient" as synonymous with "fraction."

You will notice that the word "factor" is not introduced here and for the following reason. It is felt that the mathematical concept of "factor" is such an important one that we should wait until the students are ready for its definition and application to the theory of prime factorization of integers and polynomials in Chapters 11-13.

If the teacher feels compelled to use "factor" at this point as a handy word to describe the numbers involved in an indicated product, he should do so with caution. Be sure that the students do not think of factors in terms of the form of a numeral. For example, avoid this kind of faulty thinking: "2 is not a factor of $2 + 4$ because $2 + 4$ does not involve the indicated operation of multiplication." Instead, encourage this kind of thinking: "2 is a factor of $2 + 4$ because there is a number, 3, such that the product of 2 and 3 is $2 + 4$." In general, the number a is a factor of b if and only if there is a number c such that $ac = b$. Later we learn why factoring is mathematically interesting only for integers or polynomials.

Note the use of quotes to indicate when the reference is to the numeral or expression rather than to the number represented. It is important to be careful about this at first. However, since good English does not always demand this kind of distinction but rather allows the context to give the meaning, we tend later to become more relaxed about it and use such forms as "the expression $3x - 4y + 7$" rather than "the expression '$3x - 4y + 7$'".

The agreement about the preference for multiplication over addition is made to facilitate the work with expressions and not as an end in itself. In certain kinds of expressions the agreement should also apply to division as well as multiplication, for example when division is written in the form $2/3$ or $2 \div 3$, rather than $\frac{2}{3}$. We prefer to avoid these forms and, in particular, to discourage the use of the symbol "$\div$".

The use of parentheses might be compared to the use of punctuation marks in the writing of English. Emphasis should be on the use of parentheses to enable us to read expressions without ambiguity and not on the technique of manipulating parentheses for their own sake.

Answers to Oral Exercises 2-1a; page 28:

Exercises 7, 8, 9, 10 may have more than one answer. For example, .5 and $\frac{1}{2}$ are both common names for one half. This term "common name" is introduced to improve on the old term "simpler name" which is often ambiguous.

1. 12
2. 3
3. 6
4. 1
5. 3
6. $\frac{1}{3}$
7. $\frac{4}{5}$
8. $1\frac{4}{5}$
9. $\frac{9}{10}$
10. $\frac{1}{2}$
11. 6
12. $2\frac{1}{2}$

13. 4 14. $\frac{1}{2}$ 15. $3\frac{1}{3}$

These are possible answers:

16. $8 + 4$ 17. $10 + 5$ 18. $\frac{1}{2} + \frac{1}{4}$ 19. $1 + 0$

3×4 5×3 $3 \times \frac{1}{4}$ 1×1

$14 - 2$ $18 - 3$ $1 - \frac{1}{4}$ $4 - 3$

$\frac{24}{2}$ $\frac{45}{3}$ $\frac{6}{8}$ $\frac{1}{1}$

Answers to Problem Set 2-1a; pages 28-29:

1. (a) 5 (d) 5 (g) $\frac{1}{2}$

(b) 15 (e) 4

(c) 5 (f) 1 (h) $2\frac{9}{10}$ or 2.9

2. Many responses are possible, such as:

(a) $9 - 4$; $\frac{12 - 2}{2}$ (e) 5×1; $3 + 2$

(b) $15 - 1$; 2×7 (f) $\frac{1}{2} + \frac{1}{2}$; $\frac{10}{10}$

(c) $1 + 1 + 1 + 1 + 9$; $\frac{39}{3}$ (g) $2 + 1$; $\frac{4 - 1 + 9}{4}$

(d) XXX + VI; $30 + 6$ (h) $1.7 - .8$; $.4 + .5$

3. Many responses are possible, such as:

(a) 5×2; $8 + 2$; $\frac{30}{3}$; $11 - 1$

(b) 35×1; $34\frac{1}{2} + \frac{1}{2}$; $\frac{35}{1}$; $100 - 65$

(c) $2 \times \frac{1}{3}$; $\frac{1}{3} + \frac{1}{3}$; $\frac{1 + 1}{2 + 1}$; $1 - \frac{1}{3}$

(d) 8×0; $0 + 0$; $\frac{2 - 2}{7}$; $\frac{1}{17} - \frac{1}{17}$

Answers to Oral Exercises 2-1b; page 31:

In Exercise 10, since only addition and subtraction are involved, the order is immaterial. The same is true of Exercise 13 because only multiplication and division are involved.

1. 17 3. -7 5. 11 7. 13

2. 19 4. 11 6. 1 8. 17

9. $2\frac{1}{2}$
10. 4
11. $1\frac{1}{2}$
12. $1\frac{1}{2}$
13. 1
14. $5\frac{1}{2}$
15. 4

Answers to Problem Set 2-1b; page 31:

1. 14
2. $21\frac{1}{2}$
3. 8
4. $\frac{7}{8}$
5. 4
6. 3
7. 7
8. 1
9. 9
10. 36; here, order would not make any difference.

The words "numeral" and "numerical phrase" denote almost the same thing. A phrase may be a more complicated expression which involves some operations; "numeral" includes all these and also the common names of numbers. We do not wish to make any fuss over this distinction, and are happy if the student learns to use the words in this way in the course of the year just by watching others use them. We introduce both because people do use both, and because a term for a numeral which involves some indicated operations is sometimes handy.

In the term "numerical phrase" the word "numerical" is not very important and is used not so much to distinguish it from a word phrase as from an open phrase (one involving one or more variables) which is coming.

The word "operations" is intended at this point to suggest the basic operations of arithmetic (multiplication, division, addition and subtraction). In some contexts it may be desirable to admit operations such as finding the square root, forming the absolute value, etc.

Answers to Oral Exercises 2-1c; page 33:

Exercises 1(f) and 1(j) suggest properties that will be discussed later and should not be overemphasized here except to mention that the order apparently is not important.

1. (a) Yes
   (b) Yes
   (c) No
   (d) Yes
   (e) No
   (f) Yes
   (g) Yes
   (h) No
   (i) No
   (j) Yes

Answers to Problem Set 2-1c; pages 34-35:

1.	(a)	17	(e)	13	(i)	8	(m)	13
	(b)	24	(f)	23	(j)	22	(n)	$\frac{20}{7}$
	(c)	133	(g)	19	(k)	7	(o)	$\frac{4}{3}$
	(d)	39	(h)	19	(l)	11	(p)	2

2. (a) $2 \times (3 + 1)$

   (b) $2 + (4 \times 3)$

   (c) $(6 \times 3) - 1$

   (d) $(12 - 1) \times 2$

3. (a) $(\frac{1}{2} \times 6) + 3$

   (b) $(2 \times 5) + (6 \times 2)$

   (c) $(2 \times 3) + (4 \times 3)$

   (d) $(3 \times 8) - 4$

4. (a) $11 \neq 21$, yes (e) $10 \neq 10$, no

   (b) $25 \neq 1$, yes (f) $\frac{1}{4} \neq \frac{1}{8}$, yes

   (c) $\frac{6}{4} \neq 3\frac{1}{3}$, yes (g) $11 \neq 15$, yes

   (d) $18 \neq 18$, no (h) $15 \neq 10$, yes

2-2. Sentences.

The words "true" and "false" for sentences seem preferable to "right" and "wrong" or "correct" and "incorrect" because the latter all imply moral judgments to many people. There is nothing illegal, immoral, or wrong in the usual sense of the word about a false sentence. The student should be encouraged to use only "true" and "false" in this context.

We have been doing two kinds of things with our sentences: We talk about sentences, and we use sentences. When we write

"$3 + 5 = 8$" is a true sentence,

we are talking about our language; when, in the course of a series of steps, we write

$3 + 5 = 8$,

we are *using* the language. Now when we talk about the language, we can perfectly well talk about a false sentence, if we find this useful. Thus, it is quite all right to say

"$3 + 5 = 10$" is a false sentence;

but it is far from all right to use the sentence

$$3 + 5 = 10$$

in the course of a proof. When we are actually *using* the language, false sentences have no place; when we are talking *about* our language, they are often very useful.

Check Your Reading

Question 8 should lead to a discussion of various mnemonic devices such as "points to the smaller number in a true sentence."

Answers to Oral Exercises 2-2; page 38:

1. False
2. True
3. True
4. False
5. True
6. True
7. False
8. True
9. True
10. False
11. True
12. False
13. False
14. True
15. False
16. False
17. True
18. False
19. False
20. False

Answers to Problem Set 2-2; pages 39-40:

1. (a) False
   (b) True
   (c) True
   (d) True
   (e) False
   (f) False
   (g) True
   (h) True
   (i) False
   (j) True
   (k) False

2. (a) $10 - (7 - 3) = 6$
   (b) $3 \times (5 + 7) = 36$
   (c) $(3 \times 5) + 7 = 22$
   (d) $3 \times (5 - 4) = 3$
   (e) $(3 \times 5) - 4 = 11$
   (f) $(3 \times 5) + (2 \times 4) = 23$
   (g) $3 \times (5 + 2) \times 4 = 84$
   (h) $(3 \times 5 + 2) \times 4 = 68$
   (i) $3 \times (5 - 2) \times 4 = 36$
   (j) $(3 \times 5) - (2 \times 4) = 7$
   (k) $(3 \times 5 - 2) \times 4 = 52$
   *(l) $(12 \times \frac{1}{2} - \frac{1}{3}) \times 9 = 51$
   *(m) $(12 \times \frac{1}{2}) - (\frac{1}{3} \times 9) = 3$
   *(n) $12 \times (\frac{1}{2} - \frac{1}{3}) \times 9 = 18$

In problem 2 both parentheses and the convention concerning order of operations are used.

3. (a) False
   (b) False
   (c) True
   (d) True
   (e) False
   (f) False
   (g) False
   (h) False
   (i) False
   (j) True

4. (a) Four plus eight is equal to ten plus five. False
   (b) Five plus seven is not equal to six plus five. True.
   (c) Thirteen is less than eighteen minus 7. False
   (d) One plus two is greater than zero. True.

2-3. A Property of the Number One.

This is the first time the student encounters the word "property" used in a mathematical sense. He will see this word often during the course and our object is to play heavily on the word to indicate a characteristic, a pattern, a behavior which a given element or operation displays. That is, a property of an object is something it has which is a distinguishing characteristic of the object.

The particular number 1, unlike all other numbers, has the peculiar property that the product of 1 and a given number is the given number. This is quite obvious to a student; thus, we begin our discussion of properties with the property which is easiest to understand. Later we shall call 1 an identity for multiplication. It is also a valuable property to have established (or accepted) when we introduce variables later in this chapter. Otherwise, we might have difficulty justifying that

$$\frac{3(n+4)}{3} \quad \text{and} \quad n+4$$

are names for the same number no matter what number n is.

For the time being we are content to find certain properties by considering many numerical examples and then state the generalization in words. In Chapter 4 we shall symbolize these properties using variables.

There may be a tendency on the part of the student to resist the use of the multiplication property of one in the exercises. He may feel that he is being asked to use a more complicated way of doing things which he already knows how to do.

It is important to point out to him that we are not trying to push a "new" method but rather to show the importance of the multiplication property of one. It is hoped that he will come to see that this property gives the justification for the various methods of simplifying expressions with which he may be familiar. Once the justification is understood it is all right, of course, for him to use short-cuts. Perhaps it can be said that one has to "earn" the right to use short-cuts. It is important to emphasize that in this section the multiplication property of one and the uses of this property are more important than the methodology involved in simplifying expressions.

Answers to Oral Exercises 2-3; page 44:

1. (a) $\frac{1}{4} \times \frac{5}{5} = \frac{5}{20}$
   (b) $\frac{1}{7} \times \frac{3}{3} = \frac{3}{21}$
   (c) $\frac{2}{11} \times \frac{4}{4} = \frac{8}{44}$
   (d) $\frac{4}{8} \times \frac{3}{3} = \frac{12}{24}$
   (e) $\frac{5}{2} \times \frac{3}{3} = \frac{15}{6}$
   (f) $\frac{7}{4} \times \frac{25}{25} = \frac{175}{100}$
   (g) $3 \times \frac{5}{5} = \frac{15}{5}$
   (h) $\frac{3}{7} \times \frac{5}{5} = \frac{15}{35}$

Answers to Problem Set 2-3; pages 44-45:

1. (a) $\frac{3}{5} \times \frac{8}{8} = \frac{24}{40}$
   (b) $\frac{5}{6} \times \frac{4}{4} = \frac{20}{24}$
   (c) $\frac{9}{10} \times \frac{8}{8} = \frac{72}{80}$
   (d) $\frac{25}{4} \times \frac{3}{3} = \frac{75}{12}$
   (e) $4 \times \frac{3}{3} = \frac{12}{3}$
   (f) $\frac{84}{12} \times \frac{3}{3} = \frac{252}{36}$
   (g) $\frac{40}{9} \times \frac{2}{2} = \frac{80}{18}$
   (h) $\frac{2}{11} \times \frac{9}{9} = \frac{18}{99}$

2. (a) $\frac{1}{4} + \frac{3}{8} = \frac{1}{4}(\frac{2}{2}) + \frac{3}{8}$
   $= \frac{2}{8} + \frac{3}{8}$
   $= \frac{5}{8}$

   (b) $\frac{1}{3} + \frac{3}{5} = \frac{1}{3}(\frac{5}{5}) + \frac{3}{5}(\frac{3}{3})$
   $= \frac{5}{15} + \frac{9}{15}$
   $= \frac{14}{15}$

(c) $\frac{10}{3} + \frac{7}{6} = \frac{10}{3}\left(\frac{2}{2}\right) + \frac{7}{6}$

$= \frac{20}{6} + \frac{7}{6}$

$= \frac{27}{6}$

$= \frac{9}{2}\left(\frac{3}{3}\right)$

$= \frac{9}{2}$

(d) $\frac{29}{28}$ (e) $\frac{13}{10}$ (f) $\frac{8}{3}$

3. (a) $\frac{9}{24} = \left(\frac{3}{3}\right) \times \frac{3}{8}$

(b) $\frac{15}{21} = \left(\frac{3}{3}\right) \times \frac{5}{7}$

(c) $\frac{4}{32} = \left(\frac{4}{4}\right) \times \frac{1}{8}$

(d) $\frac{14}{35} = \left(\frac{7}{7}\right) \times \frac{2}{5}$

(e) $\frac{5(3 + 4)}{5} = \left(\frac{5}{5}\right) \times (3 + 4)$

(f) $\frac{4(7 - 2)}{12} = \left(\frac{4}{4}\right) \times \frac{7 - 2}{3}$

(g) $\frac{9(8 + 3)}{15} = \left(\frac{3}{3}\right) \times \frac{3(8 + 3)}{5}$

(h) $\frac{4(6 - 5)}{4} = \left(\frac{4}{4}\right) \times (6 - 5)$

2-4. Some Properties of Addition and Multiplication.

The aim of this and the next section is to look at the fundamental properties of addition and multiplication in terms of specific numbers. We go as far as obtaining a general statement of the properties in English. You should not state the properties at this time using variables. We do not need these formulations at this point and prefer to lead up to variables in a different way in Section 2-6. It is important to emphasize the pattern idea here and you may want to do this by writing something like the following on the board when discussing, for example, the associative property for addition:

(first number + second number) + third number =
first number + (second number + third number).

The use of the properties of addition and multiplication as an aid to computation in certain kinds of arithmetic problems is both interesting and important but is not the main point of these properties. These properties will play much more fundamental a role in this course. They constitute the foundation on which the entire subject of algebra is built.

The properties will be returned to in Chapter 4 and subsequent chapters where the general statements using variables will be given. They are discussed here not only as a part of the "spiral method" but because the distributive property is used in introducing the concept of variable.

From the mathematician's point of view the statement that an operation is a binary operation on a set of elements implies that the operation can be applied to *every pair* of elements in the set. In this section we use the word *binary* only to bring out the fact that *the operation in question is applied to two elements*. We do not concern ourselves here with the question whether the operation can be applied to *every pair* of elements that can be chosen.

Each of the following five numerals

$$4 + \big(6 + (3 + 8)\big), \quad 4 + \big((6 + 3) + 8\big),$$
$$(4 + 6) + (3 + 8), \quad \big((4 + 6) + 3\big) + 8,$$
$$\big(4 + (6 + 3)\big) + 8$$

is an indicated sum of *two* numbers and each names the same number. This latter fact enables us to write "4 + 6 + 3 + 8" without any ambiguity. The fact remains, however, that addition is a binary operation.

Answers to Oral Exercises 2-4a; page 47:

1. $(4 + 2) + 7 = 6 + 7 = 13$ $\qquad$ $4 + (2 + 7) = 4 + 9 = 13$
2. $(6 + 5) + 3 = 11 + 3 = 14$ $\qquad$ $6 + (5 + 3) = 6 + 8 = 14$
3. $(2 + 9) + 4 = 11 + 4 = 15$ $\qquad$ $2 + (9 + 4) = 2 + 13 = 15$
4. $(5 + 6) + 1 = 11 + 1 = 12$ $\qquad$ $5 + (6 + 1) = 5 + 7 = 12$

5. $(\frac{1}{2} + \frac{3}{4}) + \frac{1}{4} = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}$ $\qquad$ $\frac{1}{2} + (\frac{3}{4} + \frac{1}{4}) = \frac{1}{2} + 1 = \frac{3}{2}$

6. $(\frac{1}{3} + \frac{2}{3}) + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}$ $\qquad$ $\frac{1}{3} + (\frac{2}{3} + \frac{1}{4}) = \frac{1}{3} + \frac{11}{12} = \frac{15}{12} = \frac{5}{4}$

7. $(2 + .25) + .75 = 2.25 + .75 = 3$ $\qquad$ $2 + (.25 + .75) = 2 + 1 = 3$

8. $(4\frac{1}{2} + \frac{1}{2}) + \frac{1}{4} = 5 + \frac{1}{4} = \frac{21}{4}$ $\qquad$ $4\frac{1}{2} + (\frac{1}{2} + \frac{1}{4}) = 4\frac{1}{2} + \frac{3}{4} = \frac{21}{4}$

Answers to Problem Set 2-4a; pages 47-48:

1. (a) $4 + (2 + 7) = (4 + 2) + 7$
   (b) $(6 + 1) + \frac{1}{2} = 6 + (1 + \frac{1}{2})$
   (c) $3 + (4 + 11) = (3 + 4) + 11$
   (d) $(5 + 1) + 6 = 5 + (1 + 6)$ or $5 + (1 + 6) = (5 + 1) + 6$
   (e) $(11 + 13) + 121 = 11 + (13 + 121)$ or $11 + (13 + 121) = (11 + 13) + 121$
   (f) $(\frac{1}{2} + \frac{1}{3}) + \frac{1}{4} = \frac{1}{2} + (\frac{1}{3} + \frac{1}{4})$ or $\frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) = (\frac{1}{2} + \frac{1}{3}) + \frac{1}{4}$

2. (a) $(\frac{1}{4} + \frac{3}{4}) + \frac{2}{3} = 1 + \frac{2}{3}$, easier. $\frac{1}{4} + (\frac{3}{4} + \frac{2}{3})$
   (b) $\frac{2}{3} + (\frac{1}{5} + \frac{4}{5}) = \frac{2}{3} + 1$, easier. $(\frac{2}{3} + \frac{1}{5}) + \frac{4}{5}$
   (c) $(\frac{1}{2} + \frac{1}{2}) + \frac{5}{6} = 1 + \frac{5}{6}$, easier. $\frac{1}{2} + (\frac{1}{2} + \frac{5}{6})$
   (d) $\frac{7}{8} + (\frac{3}{4} + \frac{1}{4}) = \frac{7}{8} + 1$, easier. $(\frac{7}{8} + \frac{3}{4}) + \frac{1}{4}$
   (e) $(2\frac{5}{8} + 1\frac{3}{8}) + 2\frac{1}{4} = 4 + 2\frac{1}{4}$, easier. $2\frac{5}{8} + (1\frac{3}{8} + 2\frac{1}{4})$
   (f) $2.7 + (13.2 + .8) = 2.7 + 14$, easier. $(2.7 + 13.2) + .8$

(g) $(\frac{7}{8} + \frac{3}{4}) + \frac{1}{8} = \frac{7}{8} + (\frac{3}{4} + \frac{1}{8})$, neither is easier. This is hinting at the commutative property of addition which is coming. Do not emphasize it unless some student wants to pursue it.

3. 179 millimeters
Yes.

Although the question is very easy to answer, the fact that the answer is "yes" depends upon the property studied in this section, as can be seen by these or similar calculations.

$32 + 71 + 76$	$76 + 71 + 32$
$(32 + 71) + 76$	$(76 + 71) + 32$
$103 + 76$	$147 + 32$
179	179

Answers to Problem Set 2-4b; pages 50-51:

Some of these problems might better be given as oral exercises.

1. True, commutative property of addition
2. True, commutative property of addition
3. False
4. True, commutative property of addition
5. True, but not because of the commutative property!
6. True, associative property of addition
7. True, commutative property of addition
8. False
9. True, both properties
10. False
11. True, neither property
12. False
13. True, commutative property of addition
14. True, commutative property of addition
15. False
16. True, both properties
17. True, commutative property applied twice
18. False
19. True, both properties

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20. One purpose of this problem is to help the students make a habit of quickly recognizing addition combinations which facilitate computation. Another and more immediate purpose is to help the pupils begin to become aware that these manipulations, which they may have taken for granted, are possible because of the associative and commutative properties of addition.

In these problems we do not ask specifically which properties are used in going from one step to the next. This is often tedious - particularly in the latter steps of the calculation. We do not insist that this be done at this time for we are more concerned with having the student recognize the usefulness of the properties than in having him pursue a thorough step-by-step reasoning process from beginning to end of the calculation.

In several parts of this problem there are variations on "the easiest way" to perform the additions. Comparison of some of these in class discussion should help fulfill the purposes of the question.

(a) The student may express his answer in a manner like this: "Add the 6 and the 4 to get 10, then 10 and 8 to make 18." This can be shown step by step in several ways, e. g.:

$6 + (8 + 4)$	$6 + (8 + 4)$
$6 + (4 + 8)$	$(8 + 4) + 6$
$(6 + 4) + 8$	$8 + (4 + 6)$
$10 + 8$	$8 + 10$
18	18

(b) $\frac{2}{5} + \frac{2}{3} + 1 + \frac{1}{3} + \frac{8}{5}$

$\frac{2}{5} + \frac{8}{5} + \frac{2}{3} + \frac{1}{3} + 1$

$(\frac{2}{5} + \frac{8}{5}) + (\frac{2}{3} + \frac{1}{3}) + 1$

$2 + 1 + 1$

$2 + (1 + 1)$

$2 + 2$

4.

page 51: 2-4

(c) $5\frac{4}{7} + 6 + 14\frac{3}{7}$

$5\frac{4}{7} + 14\frac{3}{7} + 6$

$(5\frac{4}{7} + 14\frac{3}{7}) + 6$

$20 + 6$

26

(d) $\frac{3}{4} + \frac{1}{3}$ This is a case in which there is no "easier" way. Neither property is of help in this computation, though several properties to be studied later, most notably the distributive property, lie behind the students' calculations.

$\frac{3}{4} + \frac{1}{3} = \frac{13}{12}$

(e) $2\frac{1}{5} + 3\frac{2}{3} + 6 + 7\frac{4}{5}$

$2\frac{1}{5} + 7\frac{4}{5} + 6 + 3\frac{2}{3}$

$(2\frac{1}{5} + 7\frac{4}{5}) + 6 + 3\frac{2}{3}$

$10 + 6 + 3\frac{2}{3}$

$(10 + 6) + 3\frac{2}{3}$

$16 + 3\frac{2}{3}$

$19\frac{2}{3}$ or $\frac{59}{3}$

(f) Here is another case in which neither property facilitates the computation. $\frac{10}{3} + \frac{6}{5} = \frac{68}{15}$

(g) $(1.8 + 2.1) + (1.6 + .9) + 1.2$

$1.2 + (1.8 + 2.1) + (.9 + 1.6)$

$(1.2 + 1.8) + (2.1 + .9) + 1.6$

$3.0 + 3.0 + 1.6$

$(3.0 + 3.0) + 1.6$

$6.0 + 1.6$

7.6

(h) $(8 + 7) + 4 + (3 + 6)$

$(8 + 7) + (3 + 6) + 4$

$8 + (7 + 3) + (6 + 4)$

$8 + 10 + 10$

$8 + (10 + 10)$

$8 + 20 = 28$

Answers to Oral Exercises 2-4c; pages 53-54:

1. True, associative property of multiplication
2. True, commutative property of multiplication
3. True, commutative property of addition
4. True, commutative property of multiplication
5. False
6. False
7. True, commutative property of addition
8. True, commutative property of addition (twice)
9. True, commutative property of multiplication
10. True, commutative property of multiplication and commutative property of addition
11. True, associative property of multiplication
12. True, associative property of addition

Answers to Problem Set 2-4c; pages 54-57:

1. This problem is intended to serve the same purposes for the properties of multiplication which Problem 20 of Problem Set 2-3b served for the addition properties.

(a) $4 \times 7 \times 25$

$4 \times 25 \times 7$

$(4 \times 25) \times 7$

100×7

700

(b)	$\frac{1}{5} \times (26 \times 5)$	$\frac{1}{5} \times (26 \times 5)$
	$\frac{1}{5} \times (5 \times 26)$	$(26 \times 5) \times \frac{1}{5}$
	$(\frac{1}{5} \times 5) \times 26$	$26 \times (5 \times \frac{1}{5})$
	1×26	26×1
	26	26

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(c) $73 + 62 + 27$
$73 + 27 + 62$
$(73 + 27) + 62$
$100 + 62$
162

This problem is a reminder that addition properties are not to be forgotten while multiplication properties are at the center of attention.

(d) $2 \times 38 \times 50$
$2 \times 50 \times 38$
$(2 \times 50) \times 38$
100×38
3800

(e) $(\frac{1}{2} \times 39) \times 2$
$(39 \times \frac{1}{2}) \times 2$
$39 \times (\frac{1}{2} \times 2)$
39×1
39

(f) $(\frac{1}{3} \times 43) \times 6$
$(43 \times \frac{1}{3}) \times 6$
$43 \times (\frac{1}{3} \times 6)$
43×2
86

(g) $\frac{1}{5} \times (18 \times 15)$
$\frac{1}{5} \times (15 \times 18)$
$(\frac{1}{5} \times 15) \times 18$
3×18
54

(h) $50 \times (97 \times 2)$
$50 \times (2 \times 97)$
$(50 \times 2) \times 97$
100×97
9700

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(i) $(\frac{3}{4} \times 19) \times 4$

$(19 \times \frac{3}{4}) \times 4$

$19 \times (\frac{3}{4} \times 4)$

19×3

57

(j) $(4 \times 8) \times (25 \times 5)$

$4 \times (8 \times (25 \times 5))$

$4 \times ((25 \times 5) \times 8)$

$4 \times (25 \times (5 \times 8))$

$(4 \times 25) \times (5 \times 8)$

100×40

4000

(k) $(3 \times 4) \times (7 \times 25)$

$3 \times (4 \times (7 \times 25))$

$3 \times ((7 \times 25) \times 4)$

$3 \times (7 \times (25 \times 4))$

$3 \times (7 \times 100)$

$(3 \times 7) \times 100$

21×100

2100

The student will probably give an answer such as "Multiply 4 times 25 and get 100; then multiply 3 times 7 and get 21; then multiply 21 times 100 and get 2100."

(l) Here is an exercise in which there is no "easiest" way, that is, regrouping is not involved.

$12 \times 14 = 168$

(m) $\frac{1}{2} \times \frac{1}{3} \times \frac{5}{6}$

$(\frac{1}{2} \times \frac{1}{3}) \times \frac{5}{6}$

$\frac{1}{6} \times \frac{5}{6}$

$\frac{5}{36}$

This way of doing the calculation is preferable only in that it involves only one digit numbers until the simplest form is written.

(n) $6 \times 8 \times 125$

$6 \times (8 \times 125)$

6×1000

6000

(o) (1.25) × 5.5 × 8
5.5 × 8 × 1.25
5.5 × (8 × 1.25)
5.5 × 10
55

(p) (2 × 5) × 1.97
10 × 1.97
19.7

Observe that in this case the original form of the problem is the best from which to work.

(q) $\frac{5}{4} \times 6 \times \frac{4}{3} \times \frac{1}{5}$
$6 \times \frac{4}{3} \times \frac{1}{5} \times \frac{5}{4}$
$6 \times \frac{4}{3} \times (\frac{1}{5} \times \frac{5}{4})$
$6 \times \frac{4}{3} \times \frac{1}{4}$
$6 \times (\frac{4}{3} \times \frac{1}{4})$
$6 \times \frac{1}{3}$
2

2. The first forms of each part of the problem are easier to compute because repetition of a partial product is involved in each case. Thus the recurring partial products can be copied after their first writing.

3. These problems are the first in which a variable occurs. It is not the intention to introduce "variable" now, but only to have the student replace "t" with the correct number. "Variable" will be discussed in Section 2-6.

(a) $t = 5$
(b) $t = 8$
(c) $t = 1\frac{1}{2}$
(d) $t = (\frac{3}{4} + 1)$ or $1\frac{3}{4}$, the commutative property of multiplication is the important part.
(e) $t = 3.7$
(f) $t = .5$
(g) $t = 7.2 + 5$ or 12.2

(h) $t = 6$. We expect some answers of $t = 4$, but this is not an example of the commutative property, since subtraction is not commutative.

(i) $t = [(3 + 2) + 5]$ or 10

(j) $t = [(7\frac{1}{2} - 2) - 5]$ or $\frac{1}{2}$

(k) $t = 4$

(l) $t = 6.2$

(m) $t = 16$. Again, $t = 4$ is wrong. Division is not commutative.

4. No. Have students give counterexample, such as

- (a) $8 \div 4 \neq 4 \div 8$
- (b) $8 - 4 \neq 4 - 8$
- (c) $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$
- (d) $(8 - 4) - 2 \neq 8 - (4 - 2)$

Problems 5 through 10 are difficult and are included only for use with the better students.

5. $2 \oplus 3 = 2 + 2(3) = 8$

$3 \oplus 2 = 3 + 2(2) = 7$

Not commutative since $2 \oplus 3 \neq 3 \oplus 2$

6. $2 \mathbin{\text{⧗}} 5 = (2 + 1) \times (5 + 1) = 18$

$5 \mathbin{\text{⧗}} 2 = (5 + 1) \times (2 + 1) = 18$

Yes, it is commutative. Don't expect the student to prove this, but he should be able to furnish several examples.

*7. $(2 \oplus 3) \oplus 4 = (2 + (2 \times 3)) \oplus 4 = 8 + (2 \times 4) = 16$

$2 \oplus (3 \oplus 4) = 2 \oplus (3 + (2 \times 4)) = 2 \oplus 11 = 2 + 2(11) = 24$

No, it is not associative.

*8. $(2 \mathbin{\text{⧗}} 5) \mathbin{\text{⧗}} 3 = ((2 + 1)(5 + 1)) \mathbin{\text{⧗}} 3 = 18 \mathbin{\text{⧗}} 3$

$= (18 + 1)(3 + 1) = 76$

$2 \mathbin{\text{⧗}} (5 \mathbin{\text{⧗}} 3) = 2 \mathbin{\text{⧗}} ((5 + 1)(3 + 1)) = 2 \mathbin{\text{⧗}} 24$

$= (2 + 1)(24 + 1) = 75$

No, it is not associative.

*9 and *10. "Keep the instructions simple" should be the caution for all except the exceptional student.

2-5. The Distributive Property.

The properties of addition and multiplication studied in the previous section appear symmetrical in form and do not really reveal anything different about the two operations. Here the student discovers from his number facts that "multiplication is distributive over addition," that is, that there is a definite connection between the operations. Although we mean "the distributive property of multiplication over addition," throughout the course we shall usually shorten this to "the distributive property." It is not necessary that the student immediately grasp the significance of the full statement of the property. An example is given to show that addition is not distributive over multiplication.

Again we use the spiral technique of presentation. One form of the distributive property, $a(b + c) = ab + ac$, is given; then after some experience with this form it is presented in the form $ab + ac = a(b + c)$. The emphasis here is on changing back and forth between indicated sums and indicated products. Later, in Chapter 4, other forms, $(b + c)a = ba + ca$, $ba + ca = (b + c)a$, are studied and used to simplify certain expressions. Even later, in Chapter 13, the distributive property is applied to the problem of multiplying polynomials and factoring polynomials. In the meanwhile many examples of the use of the property are scattered throughout the exercises.

Answers to Oral Exercises 2-5a; pages 60-61:

1. True, this does illustrate the distributive property.
2. True, this does illustrate the distributive property.
3. True, this does illustrate the distributive property.
4. False
5. False This, in fact, illustrates that addition is not distributive over multiplication.
6. True, this does not illustrate the distributive property.
7. True, this does illustrate the distributive property.
8. Indicated product
9. Indicated sum
10. Indicated sum
11. Indicated product
12. Indicated sum
13. Indicated product

Answers to Problem Set 2-5a; pages 61-62:

1. $6(8 + 4) = 6(8) + 6(4)$
2. $9(7 + 6) = 9(7) + 9(6)$
3. $0(8 + 9) = 0(8) + 0(9)$
4. $9(8 + 11) = 9(8) + 9(11)$
5. $5(8 + 4) = 5(8) + 5(4)$
6. $7(2 + 8) = 7(2) + 7(8)$
7. $3(80 + 3) = 3(80) + 3(3)$
8. $4(100 + 7) = 4(100) + 4(7)$
9. $13(10 + 1) = 13(10) + 13(1)$
10. $18(20 + 2) = 18(20) + 18(2)$

11. Not true
12. Not true
13. True
14. True. Distributive property is used.
15. Not true
16. Yes. Distributive property is used.
17. Yes. Distributive property is used.
18. Not true

19. $7(33) = 7(30 + 3)$
    $= 7(30) + 7(3)$
    $= 210 + 21$
20. $6(109) = 6(100 + 9)$
    $= 6(100) + 6(9)$
    $= 600 + 54$
21. $13(21) = 13(20) + 13(1)$
    $= 260 + 13$
22. $8(13) = 8(10) + 8(3)$
    $= 80 + 24$
23. $14(16) = 14(10) + 14(6)$
    $= 140 + 84$
24. $15(23) = 15(20) + 15(3)$
    $= 300 + 45$

Answers to Oral Exercises 2-5b; page 63:

1. $5(7 + \frac{1}{5}) = 5(7) + 5(\frac{1}{5})$
   $= 35 + 1$
2. $4(8 + \frac{1}{2}) = 4(8) + 4(\frac{1}{2})$
   $= 32 + 2$
3. $12(2 + \frac{1}{2}) = 12(2) + 12(\frac{1}{2})$
   $= 24 + 6$
4. $6(5 + \frac{1}{3}) = 6(5) + 6(\frac{1}{3})$
   $= 30 + 2$

5. $4(6 + \frac{3}{4}) = 4(6) + 4(\frac{3}{4})$
$= 24 + 3$

6. $6(\frac{1}{2} + \frac{1}{3}) = 6(\frac{1}{2}) + 6(\frac{1}{3})$
$= 3 + 2$

Answers to Oral Exercises 2-5c; page 65:

1. $2(3 + 5)$
2. $18(3.2 + .8)$
3. $(3.1)(7 + 3)$
4. $6(19.2 + .8)$
5. $3(37 + 3)$
6. $\frac{1}{2}(5 + 3)$
7. $\frac{2}{3}(8 + 4)$
8. $14(.6 + .4)$
9. $9(76 + 4)$

Answers to Problem Set 2-5c; pages 65-66:

1. $110(100) = 11{,}000$
2. $12(\frac{1}{3}) + 12(\frac{1}{4}) = 7$
3. $27(1) = 27$
4. $\frac{1}{5}(1) = \frac{1}{5}$
5. $3(1) = 3$
6. $6(\frac{2}{3}) + 6(\frac{3}{2}) = 13$
7. $9(20) = 180$
8. $100(100) = 10{,}000$
9. $5(\frac{1}{5}) + 5(\frac{4}{3}) = 7\frac{2}{3}$
10. $7(8) + 7(\frac{4}{7}) = 60$
11. $16 + 40 = 56$
12. $0(17 + 83) = 0$
13. $88(200) + 88(1) = 17{,}688$
14. $\frac{8}{9}(9) = 8$
15. $9(1) = 9$
16. $7(4) = 28$
17. $8(100) = 800$
18. $(7(\frac{1}{3}) + 7(\frac{2}{3})) + 7(5)$
$7(1) + 7(5)$
$7(6) = 42$

19. $8(10) + 8(3) = 80 + 24$
$= 104$

20. $7(100) + 7(8) = 700 + 56$
$= 756$

21. $12(10) + 12(3) = 120 + 36$
$= 156$

22. $12(20) + 12(4) = 240 + 48$
$= 288$

23. $25(10) + 25(4) = 250 + 100$
$= 350$

24. $80(10) + 80(2) = 800 + 160$
$= 960$

25. $75(1000) + 75(1)$
$= 75{,}000 + 75$
$= 75{,}075$

26. $4(6) + 4(\frac{1}{2}) = 24 + 2$
$= 26$

pages 65-66: 2-5

27. $9(8) + 9(\frac{1}{3}) = 72 + 3$
$= 75$

28. $18(1) + 18(\frac{2}{9}) = 18 + 4$
$= 22$

29. $\frac{9}{17}(1) + \frac{9}{17}(\frac{1}{9}) = \frac{9}{17} + \frac{1}{17}$
$= \frac{10}{17}$

30. $13(2000) + 13(2) = 26,000 + 26$
$= 26,026$

31. $30(50) + 30(2) = 1500 + 60$
$= 1560$

32. $101(100) + 101(1) = 10,100 + 101$
$= 10,201$

33. 21
34. 21
35. 22

36. $2(5 + 6)$
37. $3(5 + 2)$
38. $7(2 + 3)$
39. $5(7 + 3)$
40. $4(9 + 3)$

41. $4(5 + 8)$
42. $5(5 + 6)$
43. $5(5 + 6)$
44. $6(3 + 4)$

*45. $(\frac{1}{2} + \frac{2}{3})(11 + 7)$
$(\frac{1}{2} + \frac{2}{3})18$
$(\frac{1}{2})18 + (\frac{2}{3})18$
$9 + 12$
21

*46. (a) $(\frac{3}{5} + \frac{2}{3})(8 + 7)$

$(\frac{3}{5} + \frac{2}{3})15$

$(\frac{3}{5})15 + (\frac{2}{3})15$

$9 + 10$

19

(b) $(\frac{2}{7} + \frac{1}{2})(10 + 4)$

$(\frac{2}{7} + \frac{1}{2})14$

$(\frac{2}{7})14 + (\frac{1}{2})14$

$4 + 7$

11

2-6. *Variables*.

The aim of this section is to acquaint the pupil with one meaning of the word *variable*. At this point we insist that "n" or "x", or whatever letter is used as the variable, must be *thought of* as the name of a definite number although we may not have very much information about that number. In some cases, such as in the example discussed in the text, the number may be unspecified because what we want to say about it is the same for every number in a given set. This is always the case when we are interested in the pattern or form of a problem rather than in the answer. In other cases the number may be unspecified because we do not know what it is at the outset but will find it out later. Variables used in this context are usually called "unknowns." In any case try to avoid the concept of a variable as something that varies over a set of numbers.

The discussion of the example would not have been changed in any essential way if we had decided to denote the chosen number by some letter other than n.

The set of numbers from which a variable may be specified is called "domain" by some, "range" by others, depending largely upon the point of view from which the variable and its set are being seen. There are points of view, then, to support the choice of either term. Since the most natural connection for many teachers to make, when a variable and its set are mentioned, is to see that variable as the "independent" variable in a function relationship, the name "domain" for its set comes easily to mind. It must be emphasized, however, that the variable need not be seen as the "independent variable" in a function relationship, but may in fact be considered as the "dependent" variable.

Answers to Oral Exercises 2-6a; page 70:

1. 9, 12, 17
2. 10, 25, 50
3. 22, 55, 110
4. 3, 12, 27
5. 3, 18, 43
6. 21, 30, 45
7. 10, 25, 50
8. 36, 69, 124
9. $\frac{1}{4}$, $\frac{5}{8}$, $\frac{5}{4}$
10. 2, $\frac{11}{4}$, 4
11. $\frac{1}{5}$, $\frac{7}{5}$, $\frac{17}{5}$
12. 6, $\frac{48}{5}$, $\frac{78}{5}$
13. $\frac{4}{5}$, $\frac{7}{2}$, $\frac{108}{13}$
14. No
15. $n + 2$
16. $n - 7$
17. $6n$
18. $\frac{n}{4}$
19. $2n + 4$
20. Five more than some number.
21. Two less than some number.
22. Four times some number.
23. Some number divided by 5.
24. Three more than twice some number.
25. Two less than three times some number.
26. Seven times the result of finding two less than some number.
27. Some number divided by 4 and the result increased by 5.
28. The product of five more than some number and two less than the original number.

Answers to Problem Set 2-6a; pages 71-72:

1. $2, 4, \frac{22}{3}$

2. $1, \frac{13}{4}, 7$

3. $\frac{5}{3}, \frac{16}{3}, \frac{117}{11}$

4. $3, \frac{15}{2}, 15$

5. $\frac{21}{5}, \frac{300}{13}, 75$

6. $\frac{3(3) + 4(5)}{2}$

$\frac{9 + 20}{2}$

$14\frac{1}{2}$

7. $7(2) - 2(5) + \frac{2(3)}{2}$

$14 - 10 + 3$

7

8. $\frac{6(5) - 3}{3} + \frac{3(5)}{2}$

$\frac{27}{3} + \frac{15}{2}$

$9 + \frac{15}{2}$

$\frac{33}{2}$

9. $(3(2) + 5)(5 - 3)$

$(6 + 5)(2)$

$(11)(2)$

22

10. $7(3)(2 + 5)$

$7(3)(7)$

147

11. $\left(3(2) - 2(3)\right)(2(5) - 10)$

$(6 - 6)(10 - 10)$

0

12. $\frac{1}{2}(2) + \frac{1}{5}(5) + \frac{2}{3}(3)$

$1 + 1 + 2$

4

13. $\dfrac{\left(\frac{7(2)}{2} + \frac{3(5)}{2}\right) - \frac{5(3)}{2}}{2}$

$\dfrac{\frac{14}{2} + \left(\frac{3(5)}{2} - \frac{3(5)}{2}\right)}{2}$

$\frac{7}{2}$

Point out the use of the associative and commutative properties in Problem 8.

14. $3n - 7$

15. $\frac{1}{2}x + 6$

Encourage students to use different letters rather than always "n" or always "x".

16. $\frac{1}{2}(y + 6)$
17. $\frac{b}{11} + 7$
18. $\frac{c}{3} - 6$
19. $2n + 3$
20. $\frac{n + 7}{12}$
21. Four more than eight times a number.
22. Four less than the quotient of twice a number divided by three.
23. The product of eight and the difference obtained by subtracting 5 from 2 times a number.
24. Take a number, subtract 5, multiply by 12, add 0, and divide by three.

The steps in the proposed example are

$$n$$
$$3n$$
$$3n + 12$$
$$\frac{3n + 12}{3} - 2.$$

The last phrase is a numeral for $n + 2$.

The student may wonder why we insist on writing $3n + 12 = 3(n + 4)$. Either method will, of course, lead to the same result. The completion of this as

$$\frac{3(n + 4)}{3} - 2$$

and the subsequent simplification lead to the numeral $n + 2$ with less computation than the first method.

Perhaps it will satisfy most students who raise the question if you point out that the first method brings out the pattern while the second method tends to obliterate the pattern.

Some teachers have found it helpful, in introducing the notion of variables to their students, to play a number game in class in addition to the material in the text. Another successful method has been to use such a game at the board.

pages 75-76: 2-6

Example: "Choose a number from some set S - such as, for instance, the whole numbers between 1 and 30 - add 3, multiply by 2, and subtract twice the number chosen."

Different pupils try the game at the board with different numbers, and always obtain 6. Others may be instructed to leave the numerals in indicated form, another may use "number" instead of a specific numeral, and yet others may use a variable like "n" or "x" from the beginning. The board may look like this:

	4	5	number	n
	7	5 + 3	number + 3	n + 3
	14	2(5 + 3)	2(number + 3)	2(n + 3)
or	14	2·5 + 6	2(number) + 6	2n + 6
	6	2·5 + 6 - 2·5	2(number) + 6 - 2(number)	2n + 6 - 2n
or	6	6	6	6

This example uses the distributive property, which the students have seen, but it also uses associativity and commutativity with one subtraction, which they have not seen. The operations with numbers are quite simple, however, and so the "2n - 2n" should really not give any trouble. It is certainly not worth making a fuss over. If the subtraction is, for some reason, likely to give trouble, the game may always be played with an example such as the one in the text which involves no subtraction.

Answers to Problem Set 2-6b; pages 75-77:

1. $2(t + 3)$
2. $\frac{2n + 5}{3}$
3. Both forms are correct. The second is found from the first by use of the associative property of multiplication.
4. $4y$
5. Neither form is correct. $2(a + b)$ and $2a + 2b$ are correct forms.
6. (a) $\frac{9}{5}C + 32 = \frac{9}{5}(100) + 32$

$$= 180 + 32$$

$$= 212$$

(b) $\frac{h(a + b)}{2} = \frac{4(6 + 8)}{2}$

$= 2(14)$

$= 28$

(c) $P(1 + rt) = 500(1 + 0.04(3))$

$= 500(1 + .12)$

$= 500 + 60$

$= 560$

(d) $\frac{r\ell - a}{r - 1} = \frac{2(48) - 4}{2 - 1} = 96 - 4$

$= 92$

(e) $\ell wh = 24(12)(5)$

$= 1440$

7. $\frac{3(n + 2) - 6}{3}$ is the final number.

$\frac{3n + 6 - 6}{3}$

$\frac{3n}{3}$

n. Yes, we get the original number.

8. $\frac{4x + 8}{2} - 4 - 2x$

$\frac{2(2x + 4)}{2} - 4 - 2x$

$2x + 4 - 4 - 2x$

0. Every answer is zero!

9. True for all values of x!
10. True for all values of x!
11. False
12. True for all values of x!
13. False. Don't be concerned about the negative result, but caution those who want to think of subtraction as being commutative.

14. False
15. True
16. True for all values of x!
17. False
18. False
19. False
20. True for all values of x!

Answers to Review Problem Set; pages 79-81:

1. Many possible answers, for example: $4 - 1$, $\frac{6}{2}$, $\frac{5 + 1}{2}$
2. A "common name" of a number is a numeral most often used to represent the number. For example, "2" is a common name of $5 - 3$, $\frac{6}{3}$, etc.
3. We do the multiplication and division first, then the addition and subtraction.
4. 27
5. (a) true
   (b) false
   (c) false
   (d) false
   (e) true
   (f) false
6. "<"
7. (a) $7 > 5$
   (b) $3 > x$
   (c) $N > M$
8. A binary operation is an operation that is applied to only two numbers at a time.
9. (a) Yes
   (b) Yes
   (c) No
   (d) Yes
   (e) Yes
   (f) No
10. (a) Associative property of addition
    (b) Neither, it is the commutative property of addition which is illustrated.
    (c) Associative property of multiplication
    (d) Neither, the commutative property of multiplication is involved if we replace "$\neq$" by "$=$". The sentence, as it stands, is false.

page 80

11. (a) $\frac{5}{6} \times (\frac{7}{7}) = \frac{35}{42}$

(b) $\frac{1}{3} \times (\frac{4}{4}) = \frac{4}{12}$

(c) $\frac{10}{12} \times \left(\frac{1\frac{1}{2}}{1\frac{1}{2}}\right) = \frac{15}{18}$

(d) $\frac{24}{32} = \frac{3}{4} \times (\frac{8}{8})$

(e) $\frac{5(x + 2)}{20} = (\frac{5}{5}) \times \frac{x + 2}{4}$

12. (a) True, commutative property of addition and commutative property of multiplication

(b) True, commutative and associative properties of addition.

(c) True, multiplication property of one

(d) False

(e) True, commutative property of multiplication and distributive property

(f) False

(g) True, multiplication property of one

(h) False

(i) True, none of the properties are involved.

13. (a) $19(\frac{7}{8} + \frac{1}{8})$

19×1

19

(b) $15(12)$

$15(10 + 2)$

$150 + 30$

180

(c) $9(\frac{2}{5}) + 9(\frac{3}{5}) + 9(29)$

$9(\frac{2}{5} + \frac{3}{5} + 29)$

$9(30)$

270

(d) $(203)(101)$

$(203)(100 + 1)$

$20300 + 203$

20503

14. A variable is a numeral which represents a definite, but unspecified, number chosen from a given set of numbers.

page 81

15. If $x = 4$,

(a) 7

(b) 30

(c) $\frac{27}{2}$

(d) 0

(e) 5, the easy way $\frac{x^2 + x}{x} = \frac{(x)(x) + (x)(1)}{x} = \frac{x(x+1)}{x}$

$$= x + 1$$

If $x = 5$,

(a) 8

(b) 37

(c) $\frac{77}{4}$

(d) $\frac{1}{9}$

(e) 6

If $x = 6$,

(a) 9

(b) 44

(c) 26

(d) $\frac{1}{5}$

(e) 7

16. $\frac{4(2n + 2)}{8} + 7$

Simplified: $\frac{(4)(2)(n + 1)}{8} + 7$

$$n + 1 + 7$$

$$\underline{n + 8}$$

The trick is to add eight to each number n.

Suggested Test Items

1. Insert parentheses in each of the following so that the resulting sentence is true:

(a) $5 \times 4 + 3 = 35$
(b) $5 \times 4 + 3 = 23$
(c) $7 \times 2 + 2 \times 3 = 84$
(d) $7 \times 2 + 2 \times 3 = 56$
(e) $7 \times 2 + 2 \times 3 = 20$
(f) $7 \times 2 + 2 \times 3 = 48$

2. State the property illustrated by each of the following true sentences:

(a) $7 \times 3 = 3 \times 7$
(b) $5(6 + 2) = 5(6) + 5(2)$
(c) $(8 \times 2) \times 3 = 8 \times (2 \times 3)$
(d) $7 + (5 + 4) = (7 + 5) + 4$
(e) $9 + (3 + 4) = (3 + 4) + 9$

3. Which of the numerals listed below are names for 6?

(a) $\frac{6 + 6}{6}$
(b) $3(1 + 1)$
(c) $3 \times 1 + 1$
(d) $\frac{1 + (1 + 1) + 15}{3}$
(e) $3(\frac{1}{3} + \frac{1}{2}) + \frac{7}{2}$

4. Which of the following sentences are true and which are false?

(a) $7 > 2 + 3$
(b) $4(5) \neq 18 + 5$
(c) $7 + (2 \times 3) = (7 \times 2) + 3$
(d) $7 + 3 < 7 \times 3$
(e) $5(2\frac{1}{2} + 1) = 5(2\frac{1}{2}) + 5$
(f) $\frac{6 - 2}{4} \neq \frac{4}{6 - 2}$

5. Show the steps in finding the simplest name for the number indicated:

(a) $\frac{5(3 + 4) - 20}{3}$
(b) $\frac{12(5 + 3)}{6} - 10$

6. Show how you would use the associative, commutative, and distributive properties to perform each of the following computations as simply as possible:

(a) $6\frac{2}{5} + (17 + 3\frac{3}{5})$
(b) $(12.8)(7) + (12.8)(3)$
(c) $(5 \times 13) \times 20$
(d) $\frac{1}{5}(\frac{7}{8}) + \frac{1}{5}(\frac{1}{8})$

7. A certain number n is multiplied by 5, then increased by 3, and this result is multiplied by 2. Which of the following open phrases describes this statement?

(a) $2 \times 5(n + 3)$ (c) $2n(5 + 3)$

(b) $2(5n + 3)$ (d) $2(5n + 15)$

8. Given that the domain of x is the set $\{0,1,2\}$, find the value of the phrase

$$x + \frac{x + 4}{2}$$

for each value of x.

9. Use the numbers 3,7, and 5 to illustrate

(a) the associative property of multiplication,

(b) the distributive property.

10. Show how the distributive property can be used to find each of the following products:

(a) $4 \times 5\frac{3}{4}$

(b) 15×1006

(c) $6 \times (\frac{1}{2} + \frac{1}{3})$

Answers to Suggested Test Items

1. (a) $5 \times (4 + 3) = 35$
   (b) $(5 \times 4) + 3 = 23$
   (c) $7 \times (2 + 2) \times 3 = 84$
   (d) $7 \times (2 + (2 \times 3)) = 56$
   (e) $(7 \times 2) + (2 \times 3) = 20$
   (f) $((7 \times 2) + 2) \times 3 = 48$

2. (a) the commutative property of multiplication
   (b) the distributive property
   (c) the associative property of multiplication
   (d) the associative property of addition
   (e) the commutative property of addition

3. The numerals in (b), (d), and (e) are names for 6.

   (a) $\frac{6 + 6}{6} = \frac{12}{6}$
   $= 2$

   (c) $3 \times 1 + 1 = 3 + 1$
   $= 4$

4. (a) true
   (b) true
   (c) false
   (d) true
   (e) true
   (f) false

5. (a)
$$\begin{aligned} \frac{5(3 + 4) - 20}{3} &= \frac{5(3) + 5(4) - 20}{3} \\ &= \frac{15 + 20 - 20}{3} \\ &= \frac{15}{3} \\ &= 5 \end{aligned}$$

   (b)
$$\begin{aligned} \frac{12(5 + 3)}{6} - 10 &= (\tfrac{12}{6})(5 + 3) - 10 \\ &= 2(5 + 3) - 10 \\ &= 16 - 10 \\ &= 6 \end{aligned}$$

6. (a) $6\frac{2}{5} + (17 + 3\frac{3}{5}) = 6\frac{2}{5} + (3\frac{3}{5} + 17)$ commutative property of addition
   $= (6\frac{2}{5} + 3\frac{3}{5}) + 17$ associative property of addition
   $= 10 + 17$
   $= 27$

(b) $(12.8)(7) + (12.8)(3) = 12.8(7 + 3)$ distributive property

$= 12.8(10)$

$= 128$

(c) $(5 \times 13) \times 20 = (13 \times 5) \times 20$ commutative property of multiplication

$= 13 \times (5 \times 20)$ associative property of multiplication

$= 13 \times 100$

$= 1300$

(d) $\frac{1}{5}(\frac{7}{8}) + \frac{1}{5}(\frac{1}{8}) = \frac{1}{5}(\frac{7}{8} + \frac{1}{8})$ distributive property

$= \frac{1}{5}(1)$

$= \frac{1}{5}$

7. (b) The phrase "2(5n + 3)" is the correct one. It is built up in the following sequence:

$n, \quad 5n, \quad 5n + 3, \quad 2(5n + 3).$

8. If $x = 0$, we get $0 + \frac{0 + 4}{2} = 2.$

If $x = 1$, we get $1 + \frac{1 + 4}{2} = 1 + \frac{5}{2}.$

If $x = 2$, we get $2 + \frac{2 + 4}{2} = 2 + 3$

$= 5.$

9. (a) $(3 \times 7) \times 5 = 3 \times (7 \times 5)$

$21 \times 5 = 3 \times 35$

(b) $3 \times (7 + 5) = (3 \times 7) + (3 \times 5)$

$3 \times 12 = 21 + 15$

10. (a) $4 \times 5\frac{3}{4} = 4 \times (5 + \frac{3}{4})$

$= (4 \times 5) + (4 \times \frac{3}{4})$

$= 20 + 3$

$= 23$

(b) $15 \times 1006 = 15 \times (1000 + 6)$

$= 15(1000) + 15(6)$

$= 15000 + 90$

$= 15090$

(c) $6 \times (\frac{1}{2} + \frac{1}{3}) = (6 \times \frac{1}{2}) + (6 \times \frac{1}{3})$

$= 3 + 2$

$= 5$

Chapter 3

OPEN SENTENCES AND TRUTH SETS

The properties of operations which were verbalized in Chapter 2 will be formalized in Chapter 4 in symbolic form. In preparation for this formalization we first enrich our vocabulary. The concept of a sentence, from Chapter 2, is enlarged in three ways: (1) We increase the variety of relations which our sentences can express, so that inequalities are included along with equations. (2) We write open sentences which involve variables, and for which the notion of a truth set becomes important. It is essential that the student consider both equations and inequalities as sentences, as objects of algebra with equal right to our attention, and as equally interesting and useful types of sentences. (3) We consider compound sentences as well as simple sentences. While not all of these concepts are immediately necessary for stating the properties of the operations on the numbers of arithmetic, it is worthwhile to introduce them together, and they will be used many times throughout the course.

Although this chapter is devoted entirely to sentences, it must be emphasized that we do not study sentences for their own sakes. As always, our main goal is the understanding of the properties of the operations, and sentences happen to be useful language devices for recording these properties. Students quickly become enamoured of the process of solving sentences. This is good, but be sure that this enthusiasm is directed beyond the mere fun of manipulating sentences. After all, sentences are only part of the language, but not the substance, of algebra.

The teacher may want to read, as a general reference for the work of this chapter, Haag, Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 2, Section 2.

3-1. Open Sentences.

The experimentation with the example "$2x + 3 = 18$" is supposed to suggest a systematic way of guessing values of the variable which will make the sentence true. The method might also suggest how one might decide whether or not all such values have been found. For example, a value of x greater than $8\frac{1}{2}$ will give a number greater than $2(8\frac{1}{2}) + 3$ and a value smaller than $8\frac{1}{2}$ will give a number less than $2(8\frac{1}{2}) + 3$. The properties of order which are suggested here will be taken up later in Chapter 9.

Answers to Oral Exercises 3-1a; page 85:

1. (a) True
   (b) True
   (c) True
   (d) True
   (e) True
   (f) True
   (g) False
   (h) False
   (i) False

2. (a) 2
   (b) $\frac{3}{2}$
   (c) 1
   (d) $\frac{3}{2}$
   (e) 3
   (f) $\frac{8}{3}$
   (g) $\frac{2}{3}$

3. (a) 3
   (b) 2
   (c) 1
   (d) 4
   (e) 3
   (f) $\frac{7}{2}$
   (g) $\frac{9}{2}$
   (h) 0
   (i) 1
   (j) $\frac{11}{3}$
   (k) $\frac{7}{2}$
   (l) $\frac{10}{3}$

Answers to Problem Set 3-1a; page 85 - 86

1. (a) 5 is not a truth number of the sentence.
   (b) 16 is a truth number.
   (c) 3 is a truth number.
   (d) 1 is a truth number.
   (e) 5 is a truth number. (The alert student may observe that 4 is also a truth number for this sentence.)

(f) 5 is not a truth number.

(g) 12 is a truth number.

2. For finding a truth number for each of these sentences, emphasize reasonable guessing procedures that "center in" on the target. Systematic solution of equations will be dealt with in later chapters.

   (a) 8
   (b) $\frac{1}{2}$
   (c) 2
   (d) $\frac{9}{2}$
   (e) $\frac{25}{4}$
   (f) 5
   (g) $\frac{8}{5}$
   (h) 3
   (i) $\frac{5}{2}$

3. (a) 5
   (b) 5
   (c) 2
   (d) 3
   (e) $\frac{7}{2}$
   (f) $\frac{21}{2}$
   (g) 4
   (h) 6
   (i) 12
   (j) 18
   (k) 0
   (l) 5
   (m) 5
   (n) 5
   (o) $\frac{7}{3}$
   (p) $\frac{11}{4}$
   (q) $\frac{3}{2}$
   (r) $\frac{58}{11}$

Answers to Oral Exercises 3-1b; page 88:

1. (a) True
   (b) False
   (c) False
   (d) False
   (e) True
   (f) True
   (g) False
   (h) True
   (i) False
   (j) True
   (k) False
   (l) True
   (m) True
   (n) False
   (o) False
   (p) False

2. (g), (i), (n)

3. (a), (b), (e), (k), (p)

4. (c), (d), (f), (h), (j), (l), (m), (o)

Answers to Problem Set 3-1b; page 89:

1. $\{5\}$
2. All numbers greater than 4
3. $\emptyset$
4. All numbers greater than 5
5. All numbers less than 8
6. $\{3\}$
7. The set of all numbers
8. $\{0\}$
9. The set of all numbers
10. The set of all numbers
11. All numbers greater than 2
12. The set of all numbers
13. $\{\frac{2}{3}\}$
14. $\{6\}$
15. $\emptyset$
16. The set of all numbers
17. All numbers less than 27

*18. All numbers less than 5

*19. The set of all numbers except zero

*20. $\{0\}$

Answers to Oral Exercises 3-1c; page 90:

1. (a) 0
   (b) 2
   (c) 5
   (d) 3
   (e) None
   (f) None
   (g) None
   (h) 3
   (i) 2
   (j) 4

2. (a) 25
   (b) 100
   (c) 36
   (d) 121
   (e) 81
   (f) 144
   (g) 64
   (h) 196
   (i) 1
   (j) 0

3.	(a)	2	(g)	14
	(b)	4	(h)	0
	(c)	8	(i)	1
	(d)	15	(j)	5
	(e)	100	(k)	3
	(f)	3		

Answers to Problem Set 3-1c; page 91:

1.	3	6.	0	11.	2,3
2.	5	7.	1	12.	1,2
3.	3	8.	0,1	*13.	$\frac{3}{4}$
4.	2	9.	7	*14.	$\frac{5}{4}$
5.	1	10.	5.1	*15.	$\frac{1}{3}$

3-2. Truth Sets of Open Sentences.

An open sentence involving one variable has a "truth set" defined as the set of numbers for which it is true. We have no need at this time to introduce a name for the set which makes a sentence *false*. The phrase "solution set" is also used for "truth set," particularly for sentences which are in the form of equations. We shall use "solution set" later, but we want the student to use "truth set" long enough to get its full significance.

Until the introduction of the real numbers in Chapter 6, when a sentence is written and no domain is specified, the domain may be inferred to be the set of numbers of arithmetic for which the given sentence has meaning. Note, however, that when the student begins to translate "word problems" into open sentences, he will sometimes find inherent in the problem, but not spelled out for him, some further limitation upon the domain. Thus the agreement specified in the text regarding the domain refers to sentences only, and should not be extended to include "word problems."

The teacher may want to take a moment of class time to be certain that the students remember clearly the set of numbers of arithmetic. This understanding can be reinforced soon (in the next section) by the graphing of this set.

Answers to Oral Exercises 3-2a; pages 93-94:

1. {0}
2. {1,2}
3. {1}
4. {0,1}
5. {2}
6. {1}
7. {2}
8. {0}
9. ∅
10. {0}
11. ∅
12. {1,2}
13. {1,2}
14. ∅
15. {2}
16. {2}
17. {2}
18. {1,2}
19. {2}
20. ∅

Answers to Problem Set 3-2a; pages 94-96:

1. (a) T = {3} F = {0, 1, 2, 4, 5, 6, 7, 8}
   (b) T = {4} F = {0, 1, 2, 3, 5, 6, 7, 8}
   (c) T = {6, 7, 8} F = {0, 1, 2, 3, 4, 5}
   (d) T = {0, 1, 2, 3, 4, 6, 7, 8} F = {5}
   (e) T = {0, 1, 2, 3, 4, 5} F = {6, 7, 8}
   (f) T = ∅ F = W

2. (a) T = {0,1,2,3,4}
   (b) T = {2}
   (c) T = {3,4}
   (d) T = ∅
   (e) T = {0,1}
   (f) T = {4}
   (g) T = ∅
   (h) T = {8}
   (i) T = {0,1,2,3,4,5,6,7}
   (j) T = {0,1,2,3,4,5,6,7}

3. (a) T = {2,3,4,5,6}
   F = {7,8,9}

   (b) T = {0}
   F = {10,20,30,40,50}

   (c) T = {3,5}
   F = {7,9,11}

   *(d) T is the set of numbers greater than 4 and less than 7. F is the set of numbers greater than or equal to 7.

   *(e) T is the set of numbers less than 7. F is the set of numbers greater than or equal to 7, but less than 10.

*(f) T is the empty set. F is the set of numbers greater than 8.

4. (a) $T = \{0,1,2,3,4\}$
   (b) $T = \{6\}$
   (c) $T = \emptyset$
   *(d) $T = \emptyset$
   *(e) $T = \{1,2,3,4,5,6\}$ or T is the set of counting numbers less than 7.
   *(f) $T = \{0,1,2,3,4,5,6\}$

5. (a) yes
   (b) yes
   (c) no
   (d) no
   (e) no
   (f) yes
   (g) no
   (h) yes
   (i) no

6. In this exercise encourage students to give a variety of examples.

   (a) Examples are $x + 5 = x + 4$; $x + 2 < x + 1$
   (b) Examples are $x = 5$; $x + 7 = 10$
   (c) Examples are $2y + 4 = 2(y + 2)$; $x + 3 = 3 + x$
   (d) Examples are $x > 5$; $3x + 2 > 14$

   This exercise might be a good one for class discussion.

7. (a) $T = \{2\}$
   (b) T = the empty set
   (c) $T = \{3,4,5\}$
   (d) $T = \{0,1\}$

8. (a) $T = \{2\}$
   (b) $T = \{\frac{3}{4}\}$
   (c) T = set of all numbers greater than $\frac{5}{2}$ and less than 5.
   (d) T = set of all numbers greater than 0 and less than 2.

   The sets in (a) and (b) are finite.

9. In connection with these exercises the teacher should bear in mind that formal methods for solution of equations and inequalities have not been developed as yet, since they depend upon properties of the real numbers to be presented in later chapters. Somewhat

systematic guesswork is the student's method and the stress should rest more upon the fact that the value in question is indeed in the truth set, than upon the device used to discover it.

(a) $\{4\frac{2}{3}\}$
(b) $\{2\}$
(c) $\{1\}$
(d) ∅
(e) $\{12\}$
(f) $\{6\}$
(g) $\{4\}$
(h) $\{3\}$
(i) $\{\frac{12}{5}\}$
(j) the set of numbers except 0.
(k) ∅

In exercises (b), (c), (j) and (k) above, the student may need to be reminded that division by zero has been excluded in the formation of rational numbers. Later it will be stressed that since zero has no reciprocal, an expression with denominator 0 does not represent any number.

Answers to Oral Exercises 3-2b; page 97:

1. $\{2,3,4,5\}$
2. $\{4,5\}$
3. $\{5\}$
4. ∅
5. $\{0,1,2,3,4,5\}$
6. $\{0\}$
7. $\{0,1,2,3,4,5\}$
8. $\{0,1,2,3,4\}$
9. $\{0,1,2,3,4,5\}$
10. $\{0,1,2,3,4,5\}$
11. ∅
12. $\{0,1,2\}$
13. $\{1,2,3,4,5\}$
14. $\{1,2,3,4,5\}$
15. $\{0,1,2\}$

Answers to Problem Set 3-2b; page 97:

1. $\{1\}$
2. The set of all numbers greater than 1
3. The set of numbers from 0 to 1 inclusive
4. The set of all numbers greater than or equal to 1
5. The set of numbers less than 1
6. ∅
7. The set of all numbers
8. The set of all numbers
9. The set of all numbers
10. The set of all numbers greater than or equal to 1
11. The set of all numbers less than or equal to 2

*12. The set of numbers $\frac{10}{7}$ or greater

*13. The set of numbers less than or equal to $\frac{1}{2}$

*14. $\{\frac{1}{2}\}$

15. $\emptyset$

3-3. Graphs of Truth Sets.

We shall soon start saying "graph of the open sentence" instead of the more clumsy but more nearly precise "graph of the truth set of the open sentence."

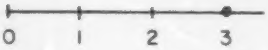
In graphing sentences whose truth set is $\emptyset$ do not fuss over the "plotting" of the empty set. Either no graph at all or a number line with no points marked is all right.

For convenience in doing problems involving the number line, you might find it helpful to duplicate sheets of number lines for the pupils' use.

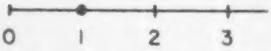
Answers to Problem Set 3-3; pages 98-99:

We have not included oral work in this section because we feel that this can readily be centered around the examples in the text, which the teacher should review carefully with the class.

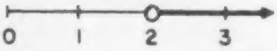
1. (a)



(b)



(c)



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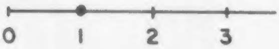
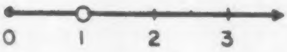
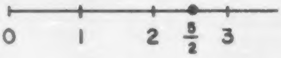


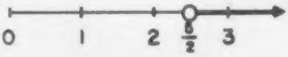
(e)



(f)



(g)



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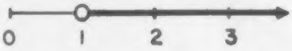
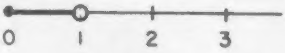
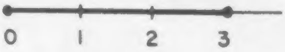


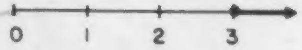
(i)



(j)



(k)



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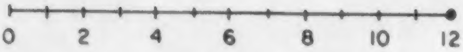
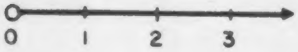
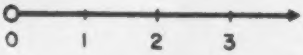


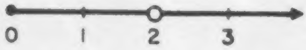
(m)



(n)



(o)



If the student shows on his graph an arrow to the left for parts (e), (i), (j), and (o), simply point out that the domain of the variable for these sentences is the numbers of arithmetic. Later when the real numbers, the operations upon them, and some of the properties of these operations are known, the student will be able to work with confidence with sentences in this extended domain.

2. (a) Yes
   (b) No
   (c) Yes--assuming that the dot on the number line has the coordinate $\frac{7}{2}$.
   (d) No, the graph is of whole numbers only--no such restriction has been placed upon x. This is a good time to re-emphasize that the domain of a variable when unspecified is the set of all numbers of arithmetic for which the sentence has meaning.
   (e) Yes

3. Accept and encourage a variety of responses in these exercises. Possible answers are:

(a) $x = 2$: $2x + 4 = 8$

(b) $x \neq 2$; $x < 2$ or $x > 2$

(c) $x \geq 4$; $3x + 5 \geq 17$

(d) $x \geq 1$; $5x + 1 \geq 6$

3-4. Compound Open Sentences and Their Graphs.

The student has dealt with simple sentences, finding their truth sets and graphing these. With compound sentences, as with simple sentences, the emphasis should be on what constitutes a truth value rather than on any technique of finding the truth values. Frequent use of the compound sentence is made throughout the course, so that further practice in this area awaits the student.

The word "clause" is used to denote a sentence which is part of a compound sentence, just as in the corresponding situation in English. The word is convenient but not very important.

Answers to Oral Exercises 3-4a; page 102:

1. Yes
   Yes
   Yes
   The left clause "$8 - 1 = 7$" and the right clause "$5 + 4 = 9$" are both true. Therefore, the compound sentence is true.
2. Yes
   No
   No
   The clause "$11 + 12 = 25$" is false. Therefore, the compound sentence "$13 - 7 = 6$ and $11 + 12 = 25$" is false.
3. No
   No
   No
   Both clauses are false. Therefore, the compound sentence "$5\frac{1}{2} + 3\frac{1}{3} = 9$ and $9 + 18 = 37$" is false.

4. (a) false
   The clause "$8 + 19 = 17$" is false.
   (b) false
   The clause "$16 \neq 8 + 8$" is false.
   (c) false
   The clause "$9 - 6 = 2$" is false.
   (d) false
   Both clauses are false.
   (e) false
   Both clauses are false.

5. (a) $\{4\}$
   (b) $\{3,4,5\}$
   (c) $\{10\}$
   (d) $\emptyset$

Answers to Problem Set 3-4a; pages 102-103:

1. (a) T, both clauses are true.
   (b) F, both clauses are false.
   (c) T, both clauses are true.
   (d) F, second clause is false.
   (e) T, both clauses are true.
   (f) F, first clause is false.
   (g) F, second clause is false.
   (h) F, second clause is false.
   (i) F, second clause is false.
   (j) F, second clause is false.
   (k) T, both clauses are true.
   (l) T, both clauses are true.

2. (a) $T = \{12\}$
   (b) $T = \{4,5,6\}$
   (c) $T = \{3\}$
   (d) $T = \emptyset$
   (e) $T = \{2\}$
   (f) $T = \emptyset$
   (g) $T = \{3,4\}$
   (h) $T = \{3\}$

The text defines the truth set of a compound sentence with the connecting word "or" as consisting of all those numbers that are in at least one of the truth sets of the clauses which make up the compound sentence. It is particularly important, not only in the interest of clarity here, but for the sake of his later work in mathematics, that the student be given a careful introduction to the phrase "at least." To have him explore such synonomous phrases as "not less than" may help pin down the idea.

Answers to Oral Exercises 3-4b; page 105:

1. Yes
   Yes
   Yes
   Both clauses are true.
2. Yes
   No
   Yes
   The first clause is true.
3. No
   No
   No
   Both clauses are false.
4. (a) true
       The first clause is true.
   (b) true
       The second clause is true.
   (c) true
       The second clause is true.
   (d) false
       Both clauses are false.
   (e) true
       Both clauses are true; the student should note, however, that since the first clause is true, the truth of the sentence is established without consideration of the second clause.

5. (a) $\{0,1,2,3\}$
   (b) $\{0,1,2,3\}$
   (c) $\{0,1,2,3,4,13\}$
   (d) $\{0,2\}$
   (e) $\{1,2\}$
   (f) $\{0,1,2\}$
   (g) $\{0,5\}$
   (h) $\{0\}$
   (i) The set of all whole numbers
   (j) The set of all whole numbers

Answers to Problem Set 3-4b; pages 105-106:

1. Each sentence which is true is true because at least one clause is true.
   (a) T
   (b) T
   (c) F, both clauses are false
   (d) T
   (e) T
   (f) F, both clauses are false
   (g) T
   (h) T
   (i) T
   (j) F, both clauses are false

2. (a) The set consisting of 5 and all numbers greater than 6
   (b) The set consisting of all numbers less than or equal to 3
   (c) The set of all numbers
   (d) $\emptyset$
   (e) The set of all numbers less than $\frac{1}{3}$ and the number 1

3. (a) True
   (b) False because "$5(8) < 5$" is false
   (c) True
   (d) False because both clauses are false
   (e) True

Answers to Problem Set 3-4c; page 108:

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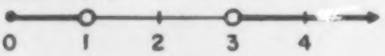

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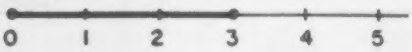

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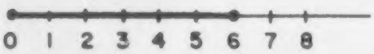

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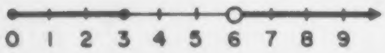
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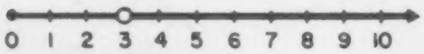
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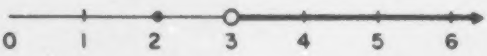

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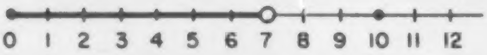
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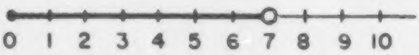
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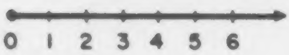
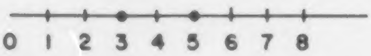


10.



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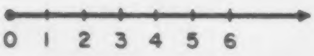


(The teacher may want to note that exercise 8 is the first graphing exercise-including examples in the text-in which the truth sets of the clauses of a compound sentence have a common value. If the student recalls the meaning of the phrase "at least," he will not find this troublesome.)

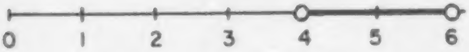
Further exercises in graphing truth sets of compound sentences with connective "or" can be obtained from the preceding section, Oral Exercises 3-4b and Problem Set 3-4b.

Answers to Oral Exercises 3-4d; page 110:

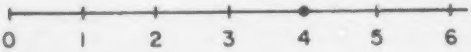
1. no
   no
2. yes
   yes
   yes
3. yes
   no
   no
4. yes
   no
   no

Answers to Problem Set 3-4d; page 110:

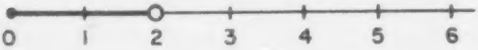
1.



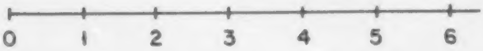
2.



3.

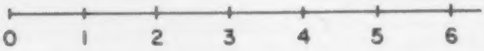


4.

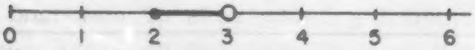


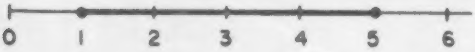
The empty set.

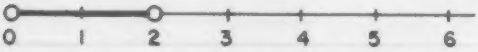
5.

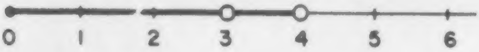


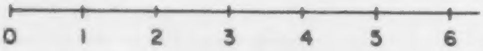
The empty set.

6. 

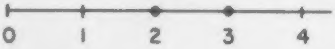
7. 

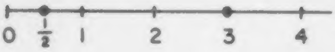
8. 

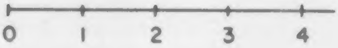
9. 

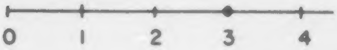
10.  The empty set.

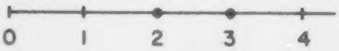
Answers to Problem Set 3-4e; page 111:

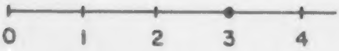
1. {2,3} 

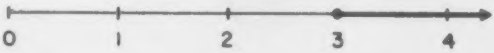
2. {$\frac{1}{2}$,3} 

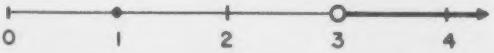
3. ∅ 

4. {3} 

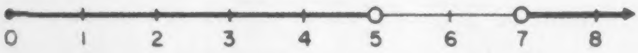
5. {2,3} 

6. {3} 

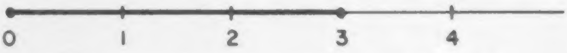
7. The set of numbers equal to or greater than 3 

8. The set which includes 1 and all numbers greater than 3 

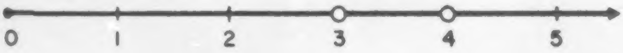
9. The set of all numbers less than 5 or greater than 7



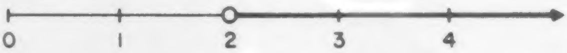
10. The set of all numbers equal to or less than 3



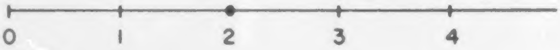
11. The set of all numbers except 3 and 4



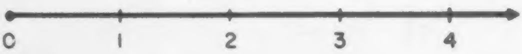
12. The set of all numbers greater than 2



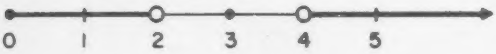
13. {2}



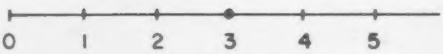
14. The set of all numbers



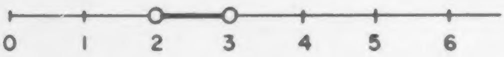
15. The set of all numbers less than two, all numbers greater than 4, and 3



16. {3}



17. The set of numbers between 2 and 3



Summary.

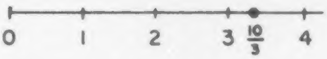
The summary is intended to help the student make a quick recall of the concepts that have been studied in the chapter.

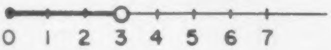
__Answers to Review Problem Set__; pages 112-115:

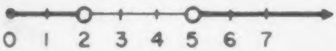
It is expected that the Review Problem Set may help the student to improve his overall understanding of mathematical sentences by giving him opportunity to work with a mixture of sentences less sorted into "types" than the problem sets throughout the chapter have been.

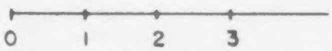
1. (a) Yes
   (b) Yes
   (c) No. $3(4) - 2 < 7$ is false
   (d) Yes
   (e) Yes (The student should note that this exercise is an instance where the commutative property of addition enables him to answer without arithmetic calculations.)
   (f) No
   (g) Yes
   (h) No
   (i) Yes
   (j) Yes

2. (a) $\{\frac{7}{2}\}$
   (b) The set of all numbers
   (c) The set of all numbers
   (d) $\{0,1\}$
   (e) $\{4\}$
   (f) $\{3\}$
   (g) $\{\frac{13}{6}\}$
   (h) The set of numbers equal to or greater than 3
   (i) The empty set
   (j) The set of all numbers greater than 2 and less than 7

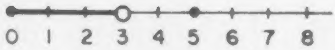
3. (a)

0
1
2
3
$\frac{10}{3}$
4

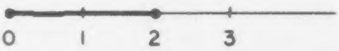
(b)


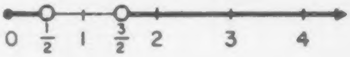
(c)


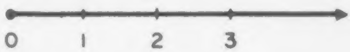
(d)


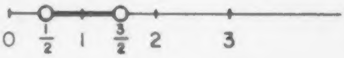
The empty set.

(e)


(f)


(g)

0
$\frac{1}{2}$
1
$\frac{3}{2}$
2
3
4

(h)


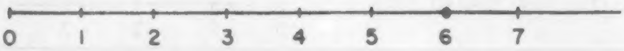
(i)

0
$\frac{1}{2}$
1
$\frac{3}{2}$
2
3

(j)

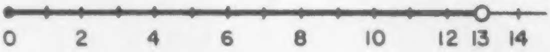
The empty set.

page 113

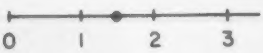
4. (a) {6}



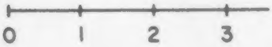
(b) The set of all numbers less than 13


0
2
4
6
8
10
12
13
14

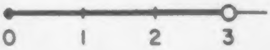
(c) $\{\frac{3}{2}\}$



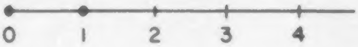
(d) The empty set



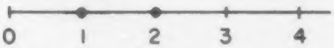
(e) The set of all numbers less than 3



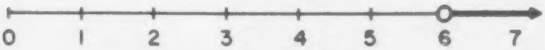
(f) {0,1}



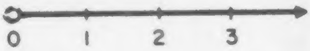
(g) {1,2}



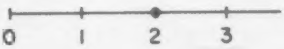
(h) The set of all numbers greater than 6



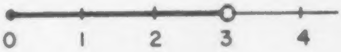
(i) The set of all numbers except 0



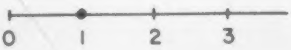
(j) {2}



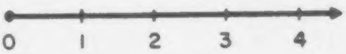
(k) The set of all numbers less than 3



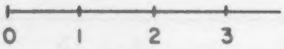
(l) {1}



(m) The set of all numbers



(n) The empty set



5. (a) True
   (b) True
   (c) False
   (d) False
   (e) False
   (f) True
   (g) False
   (h) True
   (i) True
   (j) False

6. (a) 11
   (b) $3\frac{1}{3}$ or $\frac{10}{3}$
   (c) 14
   (d) $2\frac{5}{6}$ or $\frac{17}{6}$
   (e) 0
   (f) 8
   (g) 2
   (h) 1
   (i) 0

7. The sentences in (a), (c), (d), and (e) are true for every value in the domain.

8. (a) False
   (b) True
   (c) True
   (d) True
   (e) True
   (f) True

9. (a) $\frac{2}{3} \times \frac{3}{3} = \frac{6}{9}$
   (b) $\frac{3}{5} \times \frac{5}{5} = \frac{15}{25}$
   (c) $\frac{5}{7} \times \frac{2}{2} = \frac{10}{14}$
   (d) $\frac{1}{8} \times \frac{7}{7} = \frac{7}{56}$
   (e) $\frac{5}{4} \times \frac{4}{4} = \frac{20}{16}$
   (f) $\frac{8}{3} \times \frac{4}{4} = \frac{32}{12}$

10. (a) 20
    (b) 4
    (c) 26
    (d) 2
    (e) 8
    (f) 0

11. (a) $3(10 + 3) = 3(10) + 3(3)$
    (b) $5(8) + 3(8) = (5 + 3)8$
    (c) $(4 + \frac{1}{2})6 = 4(6) + \frac{1}{2}(6)$
    (d) $3a + 5a = (3 + 5)a$
    (e) $5a + 5b = 5(a + b)$
    (f) $a(6 + b) = a(6) + ab$

Suggested Test Items

1. Which of the following sentences are true and which are false?
   (a) $5 \leq 6 + 1$
   (b) $9 + 11 \leq 5$
   (c) $8 + 2 > 3$
   (d) $5 + 3 \neq 7$

2. Which of the following sentences are true and which are false?
   (a) $\frac{6 - 3}{3} > \frac{3}{2}$ and $6 \leq 2 - 1$
   (b) $6 \neq 5 + 1$ or $4 \leq 3$
   (c) $6 + 1 = 4 + 3$ or $6 \geq 7$
   (d) $\frac{4}{7} + \frac{3}{4} > \frac{3}{5} + \frac{5}{7}$ and $4 \geq 5.3$

3. Which of the following sentences are true and which are false?

   (a) $(18 - 10) - 4 = 18 - (10 - 4)$
   (b) $(18 - 10) - 4 \neq 18 - (10 + 4)$
   (c) $3 + 4 < 8$ or $6 + 5 > 5 + 6$
   (d) $7 + 0 = 7$ and $7(0) = 7$
   (e) $4 > 6$ or $5 + 2 = 10$
   (f) $7 \geq 3$ or $17.813 + .529 = 8.777 + 18.442$

4. Determine whether each sentence is true for the given value of the variable.
   (a) $3t + 4 = 15$; 2
   (b) $4x - 3 < 7$; 7
   (c) $20 - 2x \geq 10$; 5
   (d) $\frac{x}{2} + \frac{1}{3} \neq \frac{6x}{12} + \frac{3}{9}$; 1

5. If the variables have the values assigned below, determine whether the sentence is true.
   (a) $3x = 4 + y$, x is 2 and y is 2
   (b) $5x < 2 + y$, x is 3 and y is 8

6. List the truth set of each of the following open sentences. The domain is the set of numbers indicated.
   (a) $x + 3 = 3x - 5$; $\{2,4,6\}$
   (b) $x^2 + 2 - 3x = 0$; $\{0,1,2,3\}$

7. Determine the truth sets of the following open sentences:

(a) $3x + 4 = 25$ (c) $2x + 3 = 2x + 5$

(b) $2x + 1 < 3$ (d) $4 + x \leq 2x + 1$

8. Draw the graphs of the truth sets of the open sentences:

(a) $x + 5 = 6$ (c) $2x \leq 7$

(b) $x + 1 > 3$ (d) $x \geq 4$

9. Draw the graphs of the truth sets of the compound open sentences:

(a) $x > 3$ and $x < 4$ (c) $x = 5$ or $x < 4$

(b) $x \leq 5$ and $x > 4$ (d) $x < 3$ and $x > 4$

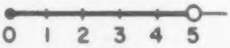
10. Which of the open sentences A, B, C, D, and E below has the same truth set as the open sentence "$p \geq 7$"?

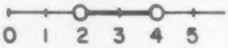
A. $p > 7$ or $p = 7$ D. $p \neq 7$

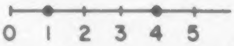
B. $p > 7$ and $p = 7$ E. $7 \leq p$

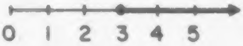
C. $p = 7$ or $p > 7$

11. Write open sentences whose truth sets are the sets graphed below:

(a)


(b)


(c)


(d)


12. For each of the sentences in column I, select the appropriate graph of its truth set in column II.

I

(a) $6x = 18$

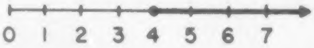
(b) $y < 3$

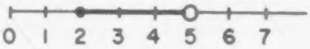
(c) $b \neq 2$

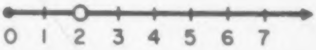
(d) $t > 4$

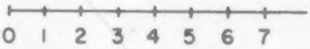
(e) $d \geq 2$ and $d < 5$

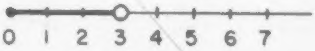
II



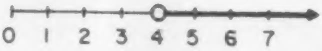




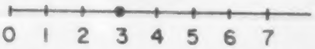




(f) $x \geq 4$



(g) $w < 2$ and $w > 4$



13. If the domain of the variable is the set $U = \{2,4,6,8,10,12\}$, find truth sets for the following open sentences:

(a) $3x + 1 = 13$

(b) $2x = 10$

(c) $2x < 20$ and $x + 4 = 4 + x$

(d) $2x + 1 = 7$ or $2x - 1 = 3$

Answers to Suggested Test Items

1. (a) True
   (b) False
   (c) True
   (d) True

2. (a) False
   (b) False
   (c) True
   (d) False

3. (a) False
   (b) False
   (c) True
   (d) False
   (e) False
   (f) True

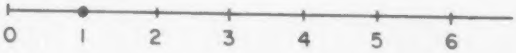
4. (a) No
   (b) No
   (c) Yes
   (d) No

5. (a) Yes
   (b) No

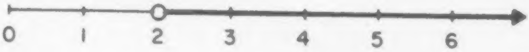
6. (a) $\{4\}$
   (b) $\{1,2\}$

7. (a) $\{7\}$
   (b) the numbers of arithmetic less than 1
   (c) $\emptyset$
   (d) the numbers greater than or equal to 3

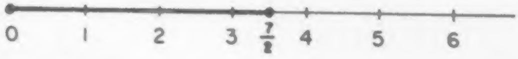
8. (a)



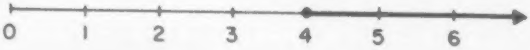
(b)



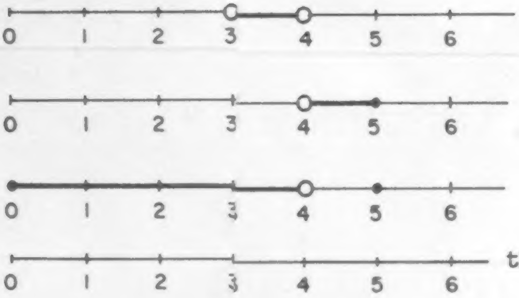
(c)



(d)



9. (a)


0 1 2 3 4 5 6
0 1 2 3 4 5 6
0 1 2 3 4 5 6
0 1 2 3 4 5 6

the empty set

10. A, C, E

11. Possible answers are:

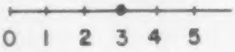
(a) $x < 5$

(b) $x > 2$ and $x < 4$

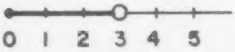
(c) $x = 1$ or $x = 4$

(d) $x \geq 3$

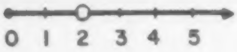
12. (a)



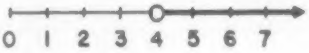
(b)



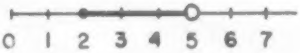
(c)



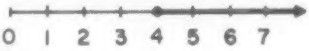
(d)



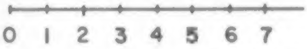
(e)



(f)



(g)



the empty set

13. (a) $\{4\}$

(b) $\emptyset$

(c) $\{2,4,6,8\}$

(d) $\{2\}$

Chapter 4

PROPERTIES OF OPERATIONS

In introducing this chapter, it is perhaps advisable for us, as teachers, to consider a basic difference between this course and the arithmetic with which the student has previously worked. The principal concern of this course is a systematic study of numbers and their properties, and arithmetic would seem to have had much the same purpose. Arithmetic often consists of a rather mechanical application of a large number of rules for computing correctly with "getting an answer" as the objective. On the other hand, we are interested in understanding rather thoroughly why numbers and operations on numbers behave as they do. A rather well defined search is made here for important general properties of the numbers and the arithmetic operations with which the student is already familiar. In short, we are interested in what is sometimes referred to as the "structure" of the "system" of numbers. Other words which convey some of the same meaning as "structure" are "pattern", "form", and "organization".

It is inevitable that many of the general properties of numbers and of the operations we apply to them are already quite familiar to the student, even the slower one, from the study of arithmetic. The properties are familiar, however, only from specific instances and not as explicit principles.

In Chapter 2, the aim was to have the student discover some of these properties by means of questions and examples. In the present chapter, the properties are studied further and are formalized. The properties which we have sought to elicit from students in this way are:

(1) Commutative and associative properties for both addition and multiplication
(2) Distributive property of multiplication over addition
(3) Addition property of 0
(4) Multiplication property of 1
(5) Multiplication property of 0

Properties (3) and (4) above state, in terms we would never use with the student before he is ready for them, that 0 and 1 are, respectively, the additive and multiplicative identities. Property (5) above is included in the list even though it can be deduced from the other properties.

It is worth noting that in this chapter we are considering the properties only in relation to the non-negative real numbers, with which the student is already familiar. We call these the numbers of arithmetic. Later, it will be seen that the same properties hold for all real numbers.

The student, conditioned as he is to arithmetic, may well ask, "Why bother?" when confronted with the formalization of these properties. This question may be forestalled somewhat by exercises which are interesting in their own right and by the teacher's own established devices. Of course, the real answer to the question "Why bother?" consists, to a large extent, of what has been said in the paragraphs preceding this one regarding our concern with structure.

Another major goal of this chapter is the development of a good deal of technique in the simplification of algebraic expressions, a conspicuous feature of any beginning algebra course. Here, however, we are introducing these techniques in conjunction with the properties of numbers and operations. Algebraic simplification is practiced at the time the principles upon which such simplification rests are first developed, and many times thereafter. These principles are precisely the properties of numbers which the student is to discover in this chapter.

The teacher may want to read, as a general reference for the work of this chapter, Haag, Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 3, Section 2.

4-1. Identity Elements.

Identity Element for Addition.

It may well be advisable to spend more time with slower students citing specific numerical instances of the addition property of zero, such as:

$$5 + 0 = 5$$

$$32\frac{1}{2} + 0 = 32\frac{1}{2}$$

$$4.7 + 0 = 4.7, \text{ etc.}$$

These may help the student appreciate the significance of the statement, "For every number a, $a + 0 = a$".

Note that the open sentence "$a + 0 = a$" is true for all values of the variable. Such a sentence conveys "structure" or "pattern" information about the number system.

The association between the "result being identical with the number to which zero is added" and the name "identity element" may be worth emphasizing. Slower students frequently need the aid of such associations in learning new words, and they are seldom successful in making the associations themselves.

Answers to Oral Exercises 4-1a; page 118:

1. {0}
2. {0}
3. {0}
4. {0}
5. {7}
6. {1}
7. the set of all numbers
8. the set of all numbers
9. the set of all numbers
10. $\emptyset$

Multiplication Property of One.

The multiplicative identity element has been introduced after the additive identity element, rather than simultaneously, in order to give the student ample time to assimilate the ideas.

The different numerals for the number one are mentioned in this section to help the student appreciate the fact that the multiplication property of one is a property of the number and has nothing to do with the numeral chosen to represent the number one. Thus,

$$"x(1) = x" \quad \text{and} \quad "x(\tfrac{4}{4}) = x"$$

both express the property.

Answers to Oral Exercises 4-1b; page 119:

1. $\{1\}$
2. $\{1\}$
3. $\emptyset$
4. $\{1\}$
5. $\{1\}$
6. the set of all numbers
7. the set of all numbers
8. $\{3\}$
9. $\emptyset$
10. $\{1\}$
11. $\{0\}$

Multiplication Property of Zero.

The multiplication property of zero is not, like the others, a fundamental property of the real number system; it would not, for example, appear among the axioms for an ordered field. It can be derived from the distributive property, the addition property of zero, and the existence of an opposite (which comes in Chapter 6). As a matter of interest, a derivation of this property is given below:

For any number a, consider the expression

$$a(1 + 0).$$

Then $a(1 + 0) = a(1) + a(0)$ by the distributive property,
but $a(1 + 0) = a(1)$ by the addition property of 0
then $a(1) + a(0) = a(1)$.

To conclude that $a(0)$ is 0, we must add $-a(1)$ (the opposite, or additive inverse of $a(1)$) to both sides of the equation above, thus obtaining

$-a(1) + \big(a(1) + a(0)\big) = -a(1) + a(1)$

then $\big(-a(1) + a(1)\big) + a(0) = -a(1) + a(1)$ by the associative property of addition,

and $0 + a(0) = 0$ by the addition property of opposites,

$a(0) = 0$ by the addition property of 0.

Answers to Oral Exercises 4-1c; page 120:

1. addition property of zero
2. multiplication property of one
3. multiplication property of zero

In Exercise 3 of Oral Exercises 4-1d and in Exercises 3, 4, 5 and 6 of Problem Set 4-1d, and in future work in which the multiplication property of 1 is used in adding fractions or rational expressions, the student should be encouraged to mention or "write out" the numeral he has used for 1. It is important that the computations which depend upon this property be clearly tied to it. Here, as at many points in the course, only a thorough appreciation of the connection between concept and manipulation entitles the student to take "short cuts." Before assigning Exercise 5 of Problem Set 4-1b you may want to remind the students that $\frac{m}{m}$ or $\frac{2b}{2b}$ is a numeral for 1 for all values of the variables except 0.

Answers to Oral Exercises 4-1d; pages 122-123:

1. (a) 6, addition property of zero
   (b) 125, multiplication property of one
   (c) 0, multiplication property of zero
   (d) $\frac{5}{3}$, addition property of zero
   (e) 2.81, multiplication property of one
   (f) 0, multiplication property of zero
   (g) 5.2, addition property of zero and multiplication property of one

2. (a) True (c) False (e) True (g) False
   (b) False (d) True (f) True (h) False

3. (a) $\frac{1}{3}(\frac{4}{4}) = \frac{4}{12}$ (e) $\frac{x}{4}(\frac{3}{3}) = \frac{3x}{12}$
   (b) $\frac{2}{9}(\frac{4}{4}) = \frac{8}{36}$ (f) $\frac{2}{3}(\frac{2a}{2a}) = \frac{4a}{6a}$
   (c) $\frac{10}{3}(\frac{3}{3}) = \frac{30}{9}$ (g) $\frac{2}{a+1}(\frac{3}{3}) = \frac{6}{3(a+1)}$
   (d) $\frac{m}{n}(\frac{3}{3}) = \frac{3m}{3n}$ (h) $\frac{4}{9}(\frac{b+3}{b+3}) = \frac{4(b+3)}{9(b+3)}$

Answers to Problem Set 4-1d; pages 123-124:

1. (a) 0. Multiplication property of zero
   (b) b. Multiplication property of one
   (c) n. Addition property of zero
   (d) n + 1. No property

2. (a) False. Numerals represent the same number because of the multiplication property of one.
   (b) True. Multiplication property of one
   (c) False. Numerals represent the same number because of the multiplication property of one.
   (d) False. The addition property of zero, i.e., for every number m, $m + 0 = m$.
   (e) False for every value of m
   (f) False for every value of m

3. (a) $\frac{18}{5} = \frac{18}{5} \times 1 = \frac{18}{5} \times \frac{13}{13} = \frac{234}{65}$
   (b) $\frac{5}{n} = \frac{5}{n} \times 1 = \frac{5}{n} \times \frac{4}{4} = \frac{20}{4n}$
   (c) $\frac{m}{x} = \frac{m}{x} \times 1 = \frac{m}{x} \times \frac{5}{5} = \frac{5m}{5x}$
   (d) $\frac{x}{10} = \frac{x}{10} \times 1 = \frac{x}{10} \times \frac{3}{3} = \frac{3x}{30}$
   (e) $\frac{2y}{5} = \frac{2y}{5} \times 1 = \frac{2y}{5} \times \frac{3}{3} = \frac{6y}{15}$
   (f) $\frac{a}{2} = \frac{a}{2} \times 1 = \frac{a}{2} \times \frac{2b}{2b} = \frac{2ab}{4b}$
   (g) $2 = 2 \times 1 = 2 \times \frac{a}{a} = \frac{2a}{a}$

4. (a) $\frac{3}{4}(\frac{2}{2}) + \frac{3}{8} = \frac{6}{8} + \frac{3}{8}$

(b) $\frac{5}{9} + \frac{1}{3}(\frac{3}{3}) = \frac{5}{9} + \frac{3}{9}$

(c) $\frac{1}{3}(\frac{4}{4}) + \frac{1}{4}(\frac{3}{3}) = \frac{4}{12} + \frac{3}{12}$

(d) $\frac{2}{\frac{1}{3}}(\frac{3}{3}) = \frac{6}{1}$

(e) $\frac{\frac{1}{5}}{3}(\frac{5}{5}) = \frac{1}{15}$

(f) $\frac{7}{\frac{2}{3}}(\frac{3}{3}) = \frac{21}{2}$

(g) $\frac{\frac{1}{3}}{6}(\frac{3}{3}) = \frac{1}{18}$

5. (a) $\frac{1}{3}(\frac{2b}{2b}) + \frac{a}{6b} = \frac{2b}{6b} + \frac{a}{6b}$

(b) $\frac{1}{2a}(\frac{3b}{3b}) + \frac{1}{3b}(\frac{2a}{2a}) = \frac{3b}{6ab} + \frac{2a}{6ab}$

(c) $\frac{\frac{1}{2}a}{3}(\frac{2}{2}) = \frac{a}{6}$

(d) $\frac{b}{\frac{1}{4}a}(\frac{4}{4}) = \frac{4b}{a}$

6. (a) $\frac{3}{7} + \frac{5}{14} = \frac{3}{7} \times 1 + \frac{5}{14} = \frac{3}{7}(\frac{2}{2}) + \frac{5}{14} = \frac{6}{14} + \frac{5}{14} = \frac{11}{14}$

(b) $\frac{5}{6} + \frac{7}{9} = \frac{5}{6} \times 1 + \frac{7}{9} \times 1 = \frac{5}{6} \times \frac{9}{9} + \frac{7}{9} \times \frac{6}{6} = \frac{45}{54} + \frac{42}{54} = \frac{87}{54}$;

or $\frac{5}{6} + \frac{7}{9} = \frac{5}{6} \times 1 + \frac{7}{9} \times 1 = \frac{5}{6} \times \frac{3}{3} + \frac{7}{9} \times \frac{2}{2} = \frac{15}{18} + \frac{14}{18} = \frac{29}{18}$

(c) $\frac{x}{3} + \frac{3}{2} = \frac{x}{3} \times 1 + \frac{3}{2} \times 1 = \frac{x}{3} \times \frac{2}{2} + \frac{3}{2} \times \frac{3}{3} = \frac{2x}{6} + \frac{9}{6} = \frac{2x + 9}{6}$

(d) $\frac{2a}{5} + \frac{b}{3} = \frac{2a}{5} \times 1 + \frac{b}{3} \times 1 = \frac{2a}{5} \times \frac{3}{3} + \frac{b}{3} \times \frac{5}{5} = \frac{6a}{15} + \frac{5b}{15} = \frac{6a+5b}{15}$

(e) $\frac{3}{\frac{1}{2}} = \frac{3}{\frac{1}{2}} \times 1 = \frac{3}{\frac{1}{2}} \times \frac{2}{2} = \frac{3 \times 2}{\frac{1}{2} \times 2} = \frac{3 \times 2}{\frac{2}{2}} = \frac{6}{1} = 6$

(f) $\frac{\frac{3}{4}}{\frac{1}{3}} = \frac{\frac{3}{4}}{\frac{1}{3}} \times 1 = \frac{\frac{3}{4}}{\frac{1}{3}} \times \frac{12}{12} = \frac{\frac{3}{4} \times 12}{\frac{1}{3} \times 12} = \frac{\frac{36}{4}}{\frac{12}{3}} = \frac{9}{4}$

(g) $\frac{a(\frac{1}{3})}{\frac{4}{1}} = \frac{(a)(\frac{1}{3})}{\frac{4}{1}} \times \frac{3}{3} = \frac{a(\frac{1}{3})(3)}{\frac{4}{1} \times 3} = \frac{a \cdot 1}{12} = \frac{a}{12}$

(h) $\frac{(b+1)(\frac{1}{5})}{\frac{3}{2}} = \frac{(b+1)(\frac{1}{5})}{\frac{3}{2}} \times \frac{10}{10} = \frac{(b+1)(\frac{1}{5})(10)}{\frac{3}{2} \times 10} = \frac{(b+1)2}{15} = \frac{2b + 2}{15}$

7. (a) The set of all numbers
   (b) {0}
   (c) {1}
   (d) {1}
   (e) The set of all numbers
   (f) {0}
   (g) {5}
   (h) {3}
   (i) {5}
   (j) {3}
   (k) {11}

4-2. Closure.

This section is concerned with two related ideas. The first is important and is one that should be impressed strongly on the student. It is this: if a is any number of arithmetic and b is any number of arithmetic, we can add a and b and we can multiply a and b. This means that we can freely write numerals such as "3a", "2b", "3a + 2b", "ab", etc. Each of these has a meaning; there is a number which it represents. Moreover, the student must be reminded over and over again that an expression such as "a + b" is a numeral rather than a command to do arithmetic. In contrast, we recall some cases in which subtraction can not be performed, and we remember that division by zero is impossible. Thus "a - b" has meaning only if a is greater than or equal to b, and "$\frac{c}{d}$" has meaning only if d is not 0.

The second idea is to introduce by examples the notion of a set of numbers being closed under an operation; a concept which the student has met informally in previous exercises. The text does not give a formal definition of this since it might be too technical. What we have in mind is this: a particular subset A of the numbers of arithmetic is closed under a particular operation (e.g., addition, subtraction, etc.) if the following statement is true: if a is any element in A and b is any element in A, the operation can be applied to a and b and, moreover, the number which is produced is an element of A. For example, the set {0} is closed under addition since $0 + 0 = 0$, it is closed under subtraction since $0 - 0 = 0$, and it is closed under multiplication since $0 \cdot 0 = 0$. It is not closed under division since division by zero cannot be done. The set {1} is closed under multiplication and under division. It is not closed under addition since $1 + 1 = 2$, and it is not closed under subtraction since $1 - 1 = 0$. Here the operations can be performed but the numbers produced are not in the set {1}. The set of even numbers {0,2,4,...} is closed under addition and under multiplication. It is not closed under subtraction and it is not closed under division. These operations cannot always be performed for this set; moreover, in some cases where they can be performed they do not yield an even number.

Answers to Oral Exercises 4-2; pages 128-129:

1. yes
yes
2. no
yes
3. yes
yes
4. yes
yes
5. (d)

Answers to Problem Set 4-2; pages 129-131:

1.	Addition	Multiplication	Division	Subtraction
(a)	closed	closed	not closed	not closed
(b)	closed	closed	not closed	not closed
(c)	closed	closed	closed	not closed
	(Can't yet subtract a larger from a smaller.)			
(d)	closed	closed	not closed	not closed
(e)	not closed	closed	not closed	not closed
(f)	closed	closed	not closed	not closed
(g)	closed	not closed	not closed	not closed
(h)	not closed	not closed	not closed	not closed

Subtraction is not closed until we extend the number system to include negative numbers, but this need not be mentioned at this time to the student. Division is not closed for any set containing 0. This again can be played down or barely mentioned at this time. Just say "We don't divide by 0" and let it go at that. Later it will be shown that 0 has no reciprocal.

2. A is not closed under addition since 2 is not in set A. A is closed under multiplication since every element in the table is an element of A.

3. Construct tables.

+	0	1	2
0	0	1	2
1	1	2	3
2	2	3	4

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	4

We see that there are elements in each table that are not elements of A. Thus, A is not closed under either operation.

4. The sets in parts (d) and (e)

5. All 4 sets

*6. C is closed under addition, but not under multiplication. Notice that these are new operations of "multiplication" and "addition" that have been defined by the table, and are not related to the operations we will be dealing with involving numbers.

*7. The operation "+" is not commutative since for example $b + a = b$ and $a + b = c$. Therefore, $a + b \neq b + a$. The operation "×" is commutative as can be shown by trying all cases, but is more readily seen by observing the symmetry of the table about the diagonal row of elements.

Class time should not ordinarily be used for discussion of the starred problems since the orderly progress of this course is not dependent upon a successful solution of such problems.

4-3. Commutative and Associative Properties of Addition and Multiplication.

The commutative and associative properties of addition and multiplication were discussed in Chapter 2 by means of numerical examples. In this section, the properties are stated in open sentences. Actually the properties are translated from word statements into the language of algebra. The translation process of which this is an example is considered more systematically in Chapter 5. Some students may find it easy to by-pass the word statement and go directly from the numerical examples to the algebraic statement of the properties. In fact, this could have been done in Chapter 2 except that we did not then have variables and so had to fall back on word statements. The comparison of the word statements with the algebra statements shows the great advantage of the latter in both clarity and simplicity.

Notice the form of the statements of the properties. If we had stated the commutative property of addition for example, without quantification of the variables as

$$a + b = b + a$$

we would have had no indication whether this open sentence is true for some, none, or all the values of a and b. Thus we quantify the variables and state: "For every number a and every number b, $a + b = b + a$." In this way we say that the open sentence is true for every a and every b.

Examples like "$(2 + 3a) + 2b = 2b + (2 + 3a)$" and "$2m + 3n = 3n + 2m$" are included (pages 131-132) since students often have trouble appreciating the generality of the statement "$a + b = b + a$." These examples are included to help head off this sort of difficulty.

The purpose of the last portion of the section is to emphasize that there are operations which are neither commutative nor associative.

We suggest that the problem sets in 4-3 be done as oral exercises with ample discussion. Perhaps this is the best way to avoid tedium and to get to the root of misunderstandings.

There seems to be no extra value to be gained by individual work on these exercises.

Answers to Oral Exercises 4-3a; page 132:

1. 3. 5. 6.

Answers to Problem Set 4-3a; pages 132-133:

1. True, commutative property of addition
2. True, commutative property of addition
3. True, commutative property of addition
4. True, commutative property of addition
5. False, since $(3a + 2b)$ may be a number different from $(3b + 2a)$

Answers to Oral Exercises 4-3b; pages 134-135:

1. (a) True, commutative property of addition
   (b) True, associative property of addition
   (c) True, associative property of multiplication
   (d) True, commutative property of multiplication
   (e) True, none of the properties
   (f) False
   (g) False
   (h) True, commutative property of multiplication
   (i) True, commutative property of addition and commutative property of multiplication
   (j) False
   (k) False
   (l) False
   (m) False
   (n) True, commutative property of addition and commutative property of multiplication

2. (a) $7(3a) = \big((7)(3)\big)a = 21a$

(b) $5m(4) = (4)(5m) = \big((4)(5)\big)m = 20m$

(c) $5(3c) = \big((5)(3)\big)c = 15c$

(d) $9(3x) = \big((9)(3)\big)x = 27x$

(e) $3x(x) = 3\big((x)(x)\big) = 3x^2$

(f) $(8y)2 = 2(8y) = \big((2)(8)\big)y = 16y$

(g) $(\frac{2}{3})(15m) = \big((\frac{2}{3})(15)\big)m = 10m$

(h) $(.1q)8 = 8(.1q) = \big((8)(.1)\big)q = .8q$

(i) $(16y)(\frac{3}{2}) = (\frac{3}{2})(16y) = \big((\frac{3}{2})(16)\big)y = 24y$

(j) $(2m)m = 2\big((m)(m)\big) = 2m^2$

(k) $(\frac{1}{2}a)(\frac{1}{4}) = (\frac{1}{4})(\frac{1}{2}a) = \big((\frac{1}{4})(\frac{1}{2})\big)(a) = \frac{1}{8}a$

*(l) $(8a)(\frac{3}{4}a) = \big((8a)(\frac{3}{4})\big)a = \Big((8)\big((a)(\frac{3}{4})\big)\Big)a$

$= \Big((8)\big((\frac{3}{4})(a)\big)\Big)a = \Big(\big((8)(\frac{3}{4})\big)a\Big)a = (6a)a$

$= 6\big((a)(a)\big) = 6a^2$

The student may not consciously go through all the steps in the exercises above, but if he is uncertain of an answer the ability to spell out the steps should reassure him.

__Answers to Problem Set 4-3c__: pages 135-136:

1. Division is not associative. $(18 \div 6) \div 2 = \frac{3}{2}$

$18 \div (6 \div 2) = 6$

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2. Subtraction is not commutative. "a - b = b - a."

   Consider $x - 3$ and $3 - x$. Whatever number of arithmetic is chosen for x, one of these expressions is not a number of arithmetic while the other is. Hence "$x - 3$" and "$3 - x$" cannot name the same number.

3. (a) True. Associative property of addition
   (b) True. Commutative property of multiplication
   (c) True. Associative and commutative properties of multiplication
   (d) True. Commutative property of multiplication
   (e) True. Commutative property of addition and commutative property of multiplication
   (f) True. Commutative property of multiplication and associative property of addition
   (g) True. Commutative and associative property of multiplication and associative property of addition
   (h) False. The left side may be written $\left(b(2) + c\right) + 2a$, which may be a number different from $\left(b(2) + c\right)2a$.
   (i) True. Commutative property of addition and commutative property of multiplication
   (j) True. Associative and commutative property of addition and commutative property of multiplication
   (k) True. Commutative and associative property of multiplication

4. (a) Yes
   (b) No for example, $(8 * 12) * 16 = 13$
   $8 * (12 * 16) = 11$

5. (a) No for example, $5 * 7 = 5$
   $7 * 5 = 7$
   (b) Yes This can be illustrated as follows:

$$(4 \Delta 6) \Delta 8 = 4 \Delta 6$$

$$= 4$$

$$4 \Delta (6 \Delta 8) = 4$$

4-4. Distributive Property.

The distributive property, like the others before it, is stated here as an open sentence, again building upon numerical experiences in Chapter 2. The property is stated in four different forms to lay the foundation for some of its future applications. However, the student should understand that there is only one distributive property under consideration.

The examples should be carefully discussed, with emphasis on the fact that these are applications of the distributive property. In example 4, you will note the phrase "simpler form". We would like to use this phrase to describe the end result. Although in most instances it is quite obvious that one form is simpler than another, it appears to be virtually impossible to give a good definition of "simple". Therefore, we will be satisfied to use the expression in concrete situations where there is no possibility of confusion and will not attempt to give a general definition. The important idea here is that, when we use the basic properties to write a phrase in another (simpler, more compact, more useful, easier to write, easier to read, etc.) form, the result is a phrase which names the same number as the given phrase.

A great deal of practice is given with the distributive property in the problem sets of Section 4-4. However, there is no need to despair if the students seem to have something less than a mastery of the principle. Following the spiral method of development, the property is used in the same and different contexts throughout future chapters; a greater degree of mastery might well await those later chapters.

Answers to Oral Exercises 4-4a; page 139:

1. (a) indicated product
   (b) indicated product
   (c) indicated sum
   (d) indicated product
   (e) indicated sum
   (f) indicated sum
   (g) indicated sum
   (h) indicated sum
   (i) indicated sum
   (j) indicated sum
   (k) indicated product
   (l) indicated sum

2. (a) $5(a) + 5(2)$
(b) $2(c) + 2(b)$
(c) $4(\frac{1}{2} + a)$
(d) $m(n) + m(1)$
(e) $5(.3 + .2)$
(f) $3(2a + 7)$
(g) $a(b + 1)$
(h) $a(b + 1)$
(i) $6(ab + 1)$
(j) $2(x + 2y)$
(k) $2(2a) + 2(3b)$
(l) $3(2a + 7)$

Answers to Oral Exercises 4-4b; page 140:

1. $4(3) + \frac{2}{3}(3)$
2. $(6 + 4)c$
3. $(2 + .3)a$
4. $7(m) + 3(m)$
5. $(7 + 8)4$
6. $9(2) + 3(2)$
7. $2(\frac{5}{2} + 4)$
8. $m(a) + 6(a)$
9. $6(a + b)$
10. $a(8) + b(8)$
11. $(a + b)3x$
12. $6(a + 1)$
13. $6(a + 1)$
14. $(3 + 1)x$
15. $(.7 + .4)m$
16. $a(b) + c(b)$
17. $5a + 5b$
18. $\frac{4}{3}(r + s)$

Answers to Problem Set 4-4b; pages 140-142:

1. (a) True
(b) True
(c) True

The aim of Exercise 1 in this problem set is to have the student recognize the truth of each sentence not because both sides can be reduced to the same common name, but because the sentence is an example of a true pattern. You may have to remind your students to do this.

2. (a) False
(b) True
(c) False
(d) False
(e) True
(f) False

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(g) True.
(h) True.
(i) True.

3. (a) $3(10) + 3(5)$
(b) $3(x) + 3(2)$
(c) $m(2) + 3(2)$
(d) $5(4) + 5(c)$
(e) $11(k) + 1(k)$
(f) $a(4) + b(4)$
(g) $ab + a(2)$
(h) $ab + ac$
(i) $x(m) + y(m)$
(j) $a(a) + 2(a)$

4. (a) $3(5 + 7)$
(b) $(3 + 7)5$
(c) $(15 + a)4$
(d) $2(b + c)$
(e) $a(2 + 5)$
(f) $(c + a)d$
(g) $a(b + 4)$
(h) cannot be done
(i) $(2 + a)a$
(j) $x(x + y)$
(k) $(4 + 3)c$
(l) $(a + 1)x$
(m) $7c$

5. The student may want to work some of these problems more quickly by "collecting terms". He may want to write "7b" immediately for part (a). Make sure that he earns the right to use these short-cuts.

(a) $5b + 2b = (5 + 2)b = 7b$
(b) $4a + a(7) = 4a + 7a = (4 + 7)a = 11a$
(c) $c(2) + c(3) = c(2 + 3) = c(5) = 5c$
(d) $\frac{1}{7}m + \frac{3}{7}m = (\frac{1}{7} + \frac{3}{7})m = \frac{4}{7}m$
(e) $.4n + .6n = (.4 + .6)n = 1n = n$
(f) $8.9b + 3.2b = (8.9 + 3.2)b = 12.1b$
(g) $3y + y = 3y + 1y = (3 + 1)y = 4y$
(h) $m + 2m = 1m + 2m = (1 + 2)m = 3m$
(i) $2a + 3b$
(j) $3.7n + n(.4) = 3.7n + .4n = (3.7 + .4)n = 4.1n$

Answers to Oral Exercises 4-4c; pages 143-144:

1. True
2. False
3. False
4. False
5. True
6. True
7. False
8. True
9. False
10. False

Answers to Problem Set 4-4c; pages 144-145:

1. (a) $6(m) + (3p)m$

(b) $(2h)k + (2h)1$

(c) $6(2s) + 6(3r)$

(d) $x(2a) + y(2a)$

(e) $7a(a) + 7a(1)$

(f) $(5a)5 + (5a)a$

(g) $2m(x) + 2m(3y)$

(h) $a(3m) + m(3m)$

(i) $x(4x) + y(4x)$

(j) $k(2x) + k(5)$

2. (a) $$\begin{aligned} 2a + 2b &= 2(a) + 2(b) \\ &= 2(a + b) \end{aligned}$$

(b) $$\begin{aligned} 2mn + 5n &= (2m)n + 5(n) \\ &= (2m + 5)n \end{aligned}$$

(c) $$\begin{aligned} 2mn + 2n &= 2nm + 2n(1) \\ &= (2n)m + (2n)1 \\ &= 2n(m + 1) \end{aligned}$$

(d) $$\begin{aligned} 6bc + 6c &= 6cb + 6c \\ &= 6c(b) + 6c(1) \\ &= 6c(b + 1) \end{aligned}$$

(e) $$\begin{aligned} 4mn + 4mp &= 4m(n) + 4m(p) \\ &= 4m(n + p) \end{aligned}$$

(f) $$\begin{aligned} cx + 4cy &= c(x) + (4c)y \\ &= c(x) + \big((c)(4)\big)y \\ &= c(x) + c(4y) \\ &= c(x + 4y) \end{aligned}$$

(g) $$\begin{aligned} 2ax + 5x &= 2a(x) + 5(x) \\ &= (2a + 5)x \end{aligned}$$

(h) $$\begin{aligned} 3ab + 9a^2 &= 3ab + (3a)(3a) \\ &= 3a(b) + (3a)(3a) \\ &= 3a(b + 3a) \end{aligned}$$

(i) $3x + 3x^2 = 3x + 3(x)(x)$
$= 3x(1) + (3x)(x)$
$= 3x(1 + x)$

(j) $xz^2 + x^2z = x(z)(z) + (x)(x)(z)$
$= (xz)(z) + x\big((x)(z)\big)$
$= (xz)(z) + xz(x)$
$= xz(z + x)$

3. (a) $3b(2 + 1)$
(b) $(2 + 3)a$
(c) $5a(x + 1)$ Note: $5a$ can be written "$5a(1)$"
(d) $a(5b + 6c)$
(e) $(m + 1)x^2$
(f) $6x(3x + 1)$

4. (a) $(3 + 2)x^2$
(b) $(a + b)x^2$
(c) $5(x^2 + c)$
(d) $3c(3b + 2)$
(e) $4b(3a + 2c)$
(f) $6a(1 + a)$

5. (a) A rectangle has two equal sides and two equal ends and so its perimeter is found by adding the number of inches in the length and the number of inches in the width and multiplying the result by 2.

$$2(7 + 3) = 2(10)$$
$$= 20$$ The perimeter is 20 inches.

(b) $1.5(375 + 125) = 1.5(500)$
$$= 750$$

The amount of money collected is $750. We could have found the amount collected at each window and then added the two amounts. This would certainly be more complicated.

Answers to Oral Exercises 4-4d; page 146:

1. $(a + c)(a + 4) = (a + c)a + (a + c)4$
2. $(x + a)(x + 3) = (x + a)x + (x + a)3$
3. $(x + 1)(a + b) = (x + 1)a + (x + 1)b$
4. $(3a + 4)(a + 5) = (3a + 4)a + (3a + 4)5$
5. $(7 + x)(x + 7) = (7 + x)x + (7 + x)7$
6. $(3a + b)(c + d) = (3a + b)c + (3a + b)d$
7. $(mn + x)(a + b) = (mn + x)a + (mn + x)b$
8. $(ab + c)(b + c) = (ab + c)b + (ab + c)c$
9. $(8 - x)(8 + x) = (8 - x)8 + (8 - x)x$

Answers to Problem Set 4-4d; pages 146 - 147:

1. $$\begin{aligned}(a + 1)(a + b) &= (a + 1)a + (a + 1)b \\ &= a^2 + a + ab + b\end{aligned}$$

2. $$\begin{aligned}(a + 5)(a + b) &= (a + 5)a + (a + 5)b \\ &= a^2 + 5a + ab + 5b\end{aligned}$$

3. $$\begin{aligned}(2x + c)(x + 4) &= (2x + c)x + (2x + c)4 \\ &= 2x^2 + cx + 8x + 4c\end{aligned}$$

4. $$\begin{aligned}(3 + m)(5 + a) &= (3 + m)5 + (3 + m)a \\ &= 15 + m(5) + 3a + ma\end{aligned}$$

5. $$\begin{aligned}(a + b)(c + d) &= (a + b)c + (a + b)d \\ &= ac + bc + ad + bd\end{aligned}$$

6. $$\begin{aligned}(2c + d)(a + d) &= (2c + d)a + (2c + d)d \\ &= 2ca + da + 2cd + d^2\end{aligned}$$

7. $$\begin{aligned}(20 + 5)(40 + 3) &= (20 + 5)40 + (20 + 5)3 \\ &= 800 + 200 + 60 + 15 \\ &= 1075\end{aligned}$$

8. $$\begin{aligned}(x + 2) + (x + 5) &= \big((x + 2) + x\big) + 5 \\ &= \big(x + (x + 2)\big) + 5 = \big((x + x) + 2\big) + 5 \\ &= \big((1x + 1x) + 2\big) + 5 = \big((1 + 1)x + 2\big) + 5 \\ &= (2x + 2) + 5 \\ &= 2x + (2 + 5) \\ &= 2x + 7\end{aligned}$$

9. $$\begin{aligned}(20 + 5)(x + 3) &= (20 + 5)x + (20 + 5)3\\ &= 20x + 5x + 60 + 15\\ &= 25x + 75\end{aligned}$$

10. $$\begin{aligned}(a + 2b)(2a + c) &= (a + 2b)\,2a + (a + 2b)c\\ &= 2a^2 + 4ab + ac + 2bc\end{aligned}$$

11. $m(m + n) = m^2 + mn$

12. $$\begin{aligned}(x + y)(m + n) &= (x + y)m + (x + y)n\\ &= xm + ym + xn + yn\end{aligned}$$

13. $$\begin{aligned}(3r + 1)(r + a) &= (3r + 1)r + (3r + 1)a\\ &= 3r^2 + r + 3ra + a\end{aligned}$$

14. $$\begin{aligned}(3r + 1)(r + 3a) &= (3r + 1)r + (3r + 1)3a\\ &= 3r^2 + r + 9ra + 3a\end{aligned}$$

15. $$\begin{aligned}(mn + b)(mn + a) &= (mn + b)mn + (mn + b)a\\ &= (mn)^2 + bmn + mna + ba\end{aligned}$$

16. $$\begin{aligned}(xy + a)(y + b) &= (xy + a)y + (xy + a)b\\ &= xy^2 + ay + xyb + ab\end{aligned}$$

17. If a is 5 and x is 2, then
$$\begin{aligned}(a + 2)(a + x) &= (5 + 2)(5 + 2)\\ &= 49\end{aligned}$$
If a is 5 and x is 2, then
$$\begin{aligned}a^2 + 2a + ax + 2x &= (5)^2 + 2(5) + (5)(2) + 2(2)\\ &= 49\end{aligned}$$
Therefore, $(a + 2)(a + x) = a^2 + 2a + ax + 2x$
when a is 5 and x is 2.

18. If x is 3 and a is 0,
$$\begin{aligned}(2x + 3)(x + a) &= (6 + 3)(3 + 0)\\ &= 27\end{aligned}$$
If x is 3 and a is 0,
$$\begin{aligned}2x^2 + 3x + 2ax + 3a &= 2(3)^2 + 3(3) + 2(0)(3) + 3(0)\\ &= 27\end{aligned}$$
Therefore, $(2x + 3)(x + a) = 2x^2 + 3x + 2ax + 3a$
when x is 3 and a is 0.

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19. (a) $(10 + 2)(20 + 4)$
(b) $(20 + 2)(20 + 1)$
(c) $(10 + 6)(10 + 2)$
(d) $(10 + 8)(60 + 1)$
(e) $(20 + 5)(30 + 2)$
(f) $(40 + 2)(30 + 6)$
(g) $(30 + 3)(20 + 3)$
(h) $(10 + 1)(4 + .2)$
(i) $(3 + \frac{1}{2})(2 + \frac{1}{2})$
(j) $(30 + 5)(30 + 5)$
(k) $(1 + .5)(30 + 6)$
(l) $(4 + .5)(4 + .5)$
(m) $(40 + 5)(200 + 2)$
(n) $(20 + 5)(1000 + 3)$
(o) $(4 + \frac{1}{3})(3 + \frac{3}{4})$
(p) $(6 + .4)(400 + 8)$
(q) $(6 + \frac{1}{4})(8 + \frac{2}{3})$
(r) $(3 + \frac{3}{4})(4 + .8)$

Summary.

This summary of properties is very important. We want the student to begin thinking of the number system more and more often in terms of the basic properties so that eventually almost all operations he does with numbers will be performed with these properties in mind. This is a development which will not take place for most students very quickly; however, by the end of the year it is hoped that the majority will have progressed to within sight of the goal.

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The list of properties obtained at this point is not complete. We still must introduce the negative numbers and obtain the properties of order. The list will be completed for our purposes by the end of Chapter 12.

Answers to Review Problem Set; pages 148-151:

1. Zero is the identity element of addition.
   For every number a, $a + 0 = a$ Addition property of zero
   For every number a, $a(0) = 0$ Multiplication property of zero

2. One is the identity element of multiplication.
   For every number a, $a(1) = a$ Multiplication property of one
   $\frac{5}{5}$, $\frac{(2)(6)}{(3)(4)}$, $\frac{m}{m}$ (m not 0) are numererals for one.

3. (a) $x + b$
   (b) a
   (c) $\frac{3}{8} + \frac{4}{3} = \frac{3}{8} \times 1 + \frac{4}{3} \times 1 = \frac{3}{8} \times \frac{3}{3} + \frac{4}{3} \times \frac{8}{8} = \frac{9}{24} + \frac{32}{24} = \frac{41}{24}$
   (d) $\dfrac{\frac{3}{8}}{\frac{4}{3}} = \dfrac{\frac{3}{8}}{\frac{4}{3}} \times 1 = \dfrac{\frac{3}{8}}{\frac{4}{3}} \times \frac{24}{24} = \dfrac{\frac{3}{8} \times 24}{\frac{4}{3} \times 24} = \dfrac{\frac{3 \times 24}{8}}{\frac{4 \times 24}{3}} = \frac{9}{32}$
   (e) $\left(\dfrac{\frac{5}{8}}{\frac{3}{4}}\right)\bigl(14(0)\bigr) = \left(\dfrac{\frac{5}{8}}{\frac{3}{4}}\right)(0) = 0$

4. (a) $\frac{1}{2} = \frac{1}{2} \times \frac{48}{48} = \frac{48}{96}$
   (b) $\frac{4}{3} = \frac{4}{3} \times \frac{15}{15} = \frac{60}{45}$
   (c) $\frac{2}{5} = \frac{2}{5} \times \frac{m}{m} = \frac{2m}{5m}$
   (d) $\frac{m}{n} = \frac{m}{n} \times \frac{5}{5} = \frac{5m}{5n}$
   (e) $\frac{5}{n} = \frac{5}{n} \times \frac{2m}{2m} = \frac{10m}{2mn}$

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5. (a) $\{0\}$
   (b) $\{1\}$
   (c) The set of all numbers
   (d) $\{0\}$
   (e) $\{1\}$
   (f) $\{0\}$
   (g) $\{1\}$
   (h) $\left\{\frac{5}{3}\right\}$
   (i) $\emptyset$
   (j) $\{0\}$

6. Set of whole numbers ending in 0 is closed under addition, is closed under multiplication, and is not closed under subtraction.

7. For every number a, every number b, and every number c

   $a + b = b + a$ $\quad$ $ab = ba$ $\quad$ Commutative properties
   $(a+b) + c = a + (b+c)$ $\quad$ $(ab)c = a(bc)$ $\quad$ Associative properties
   $a(b + c) = ab + ac$ $\quad$ Distributive property

8. (a) False. If x is 1, m is 2, y is 3, and n is 4, then $(1 + 3)(2 + 4) = (1 + 2)(3 + 4)$ is a false sentence.
   (b) True. Commutative property of addition
   (c) True. Commutative property of addition
   (d) True. Commutative property of multiplication
   (e) False. If x is 2, y is 3, and m is 1, then $2 + 3(1) = 2(3) + 1$ is a false sentence.
   (f) False. If x is 2, y is 3, and m is 1, then $2(3 + 1) = 3(2 + 1)$ is a false sentence.
   (g) True. Commutative property of multiplication
   (h) True. Distributive property and commutative property of multiplication
   (i) False. If x is 2, y is 3, and m is 1, $2 + 3(1) = (2 + 3)(2 + 1)$ is a false sentence.
   (j) True. Associative property of multiplication

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9. (a) $2n + 5$

(b) $\frac{1}{3}n - 6$ or $\frac{n}{3} - 6$

(c) $3n - 6$

(d) $n(5 + n)$

(e) $n^2 + 2n$

(f) $(n + 5)(n + 2)$

(g) $(n + 7)^2$

(h) $n(n - 2)$

(i) $\frac{n}{8}$

(j) $\frac{n + 6}{n}$

10. (a) 0

(b) $\frac{2}{5}$

(c) 0

(d) 0

(e) 7

How hard did the students work on these? Are they using the properties of 0 and 1 to make their work easy?

11. (a) $3m + 3n$

(b) $6(m + n)$

(c) $ab + ac$

(d) $xb + xc + yb + yc$

(e) $7xy + 7x$

(f) $a^2x + am$

(g) $a(ab + c)$

(h) $axy + 8xy$

(i) $4a^2 + a + 4ra + r$

(j) $ab + ax + 3b + 3x$

(k) $(3 + a)(x + y)$

(l) $6a(x + y)$

(m) $4y^2(2 + 1)$

(n) $6(2x + 3)$

(o) $a^2b + ab^2$

(p) $x(y + z)$

(q) $6a^2 + 10a$

(r) $3m^2 + 3mn$

(s) $a^2 + ac + 3ba + 3bc$

(t) $abc(x + y)$

(u) $6(a + b)(3 + 1)$

(v) $(x + y)(m + n)$

12. (a)
$$\begin{aligned} 3 \times \left(2 + \frac{3}{7}\right) &= (3 \times 2) + \left(3 \times \frac{3}{7}\right) \\ &= 6 + \frac{9}{7} \\ &= 7\frac{2}{7} \end{aligned}$$

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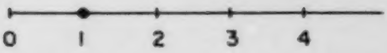
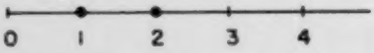
(b) $$\begin{aligned} 5 \times 4\tfrac{2}{13} &= 5 \times (4 + \tfrac{2}{13}) \\ &= (5 \times 4) + (5 \times \tfrac{2}{13}) \\ &= 20 + \tfrac{10}{13} \\ &= 20\tfrac{10}{13} \end{aligned}$$

(c) $$\begin{aligned} 4 \times 1\tfrac{5}{8} &= 4 \times (1 + \tfrac{5}{8}) \\ &= (4 \times 1) + (4 \times \tfrac{5}{8}) \\ &= 4 + \tfrac{5}{2} \\ &= 6\tfrac{1}{2} \end{aligned}$$

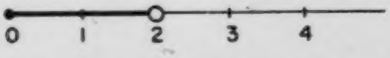
13. truth set: $\{1\}$ truth set: $\{1,2\}$

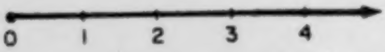




The truth set of the first sentence is a subset of the truth set of the second sentence.

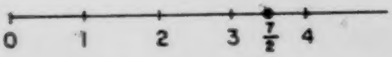
14. truth set: All numbers less than 2 truth set: All numbers

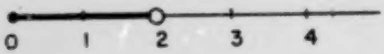




The first truth set is a subset of the second truth set.

15. truth set: $\{\frac{7}{2}\}$ truth set: All numbers less than 2





Neither is a subset of the other.

Suggested Test Items

1. Show how to use the multiplication property of 1 to find common names for

   (a) $\frac{2}{3} + \frac{3}{4}$

   (b) $\dfrac{2 + \frac{1}{3}}{5}$

   (c) $\dfrac{1 + \frac{1}{3}}{1 - \frac{1}{3}}$

2. Which of the following sentences are true for every value of the variables? (Give reasons for your answers.)

   (a) $x(2 + 3) = (2 + 3)x$
   (b) $b(a + 2) = a(b + 2)$
   (c) $a^2b^2 = b^2a^2$
   (d) $(4x + y)3 = 4x(y + 3)$
   (e) $(3a + c) + d = (c + d) + 3a$
   (f) $(2x)y = 2(xy)$
   (g) $a(b - b) = a$
   (h) $a + (b - b) = a$

3. Each sentence below is true for every value of the variables. In each case decide which properties enable us to verify this fact.

   (a) $x(y + z) = xy + xz$
   (b) $xy + (ay + c) = (xy + ay) + c$
   (c) $abcd = ab(cd)$
   (d) $xy + xz = yx + zx$
   (e) $(ab)(cd) = (dc)(ba)$
   (f) $x + 0 = x$
   (g) $0(x) = 0$
   (h) $1(x) = x$

4. Show how it is possible to use the distributive property to find common names for the following in an easy way.

   (a) $(212)(101)$
   (b) $(37)(\frac{5}{3}) + (37)(\frac{4}{3})$
   (c) $60(\frac{5}{6} + \frac{3}{5})$
   (d) $(13)(29)$
   (e) $(\frac{2}{9} + \frac{5}{6})(72)$
   (f) $(1.4)(43) + (1.6)(43)$

5. For each step except the last in the following, state which property of the operations is used to derive it from the preceding step.

$$\begin{aligned} 3(2x + y) + 5x &= (6x + 3y) + 5x \\ &= (3y + 6x) + 5x \\ &= 3y + (6x + 5x) \\ &= 3y + (6 + 5)x \\ &= 3y + 11x \end{aligned}$$

6. Use the properties of the operations to write the following open phrases in simpler form.
   - (a) $6x + 3x$
   - (b) $41a + 37b + 82a + 14b$
   - (c) $.3x + 1.4y + 7.1x + 1.1z + 2.3y$
   - (d) $\frac{5}{4} + \frac{2}{7}x + \frac{16}{5}y + \frac{5}{7}x + \frac{4}{5}y$

7. Find the truth sets of the following sentences.
   - (a) $4x = 0$
   - (b) $4y = 4$
   - (c) $\frac{4}{4}z = 1$
   - (d) $4(a + 4) = 12$
   - (e) $v + 2 = \frac{4}{4}v + \frac{4}{4}$
   - (f) $(4 - 4)w = (w - w)4$

8. Change indicated products to indicated sums, and indicated sums to indicated products, using the distributive property.
   - (a) $5x + 5y$
   - (b) $(u + 2v)4$
   - (c) $3a(2 + 4b)$
   - (d) $2(a + 2) + x(a + 2)$
   - (e) $(a + 2)(b + 1)$
   - (f) $(x + y)(x + 1)$

Answers to Suggested Test Items

1. (a) $\frac{2}{3} + \frac{3}{4} = \frac{2}{3} \times 1 + \frac{3}{4} \times 1 = \frac{2}{3} \times \frac{4}{4} + \frac{3}{4} \times \frac{3}{3} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$

(b)

$$\frac{2 + \frac{1}{3}}{5} = \left(\frac{2 + \frac{1}{3}}{5}\right)(1) = \left(\frac{2 + \frac{1}{3}}{5}\right)\left(\frac{3}{3}\right) = \frac{(2 + \frac{1}{3})(3)}{15}$$

$$= \frac{6 + 1}{15} = \frac{7}{15}$$

(c)

$$\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = \left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right)(1) = \left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right)\left(\frac{3}{3}\right) = \frac{(1+\frac{1}{3})(3)}{(1-\frac{1}{3})(3)} = \frac{3 + 1}{3 - 1}$$

$$= \frac{4}{2} = 2$$

or

$$\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = \left(\frac{\frac{4}{3}}{\frac{2}{3}}\right)(1) = \left(\frac{\frac{4}{3}}{\frac{2}{3}}\right)\left(\frac{3}{3}\right) = \frac{(\frac{4}{3})(3)}{(\frac{2}{3})(3)} = \frac{4}{2} = 2$$

2. (a) True. Commutative property of multiplication

(b) False. $ba + b(2)$ and $ab + a(2)$ are different numbers if $a \neq b$.

(c) True. Commutative property of multiplication

(d) False. $12x + 3y$ and $4xy + 12x$ are different numbers unless $3y$ is $4xy$, i.e., unless y is 0 or x is $\frac{3}{4}$.

(e) True. Associative and commutative properties of addition

(f) True. Associative property of multiplication

(g) False. By the multiplication of zero, for any number a, $a(0) = 0$.

(h) True. Addition property of zero

3. (a) Distributive property

(b) Associative property of addition

(c) Associative property of multiplication

(d) Commutative property of multiplication

(e) Commutative property of multiplication

(f) Addition property of zero

(g) Multiplication property of zero

(h) Multiplication property of one

4. (a) $(212)(101) = 212(100 + 1) = 21{,}200 + 212 = 21{,}412$

(b) $(37)(\frac{5}{3}) + (37)(\frac{4}{3}) = (37)(\frac{5}{3} + \frac{4}{3}) = (37)(\frac{9}{3}) = (37)(3) = 111$

(c) $60(\frac{5}{6} + \frac{3}{5}) = (60)(\frac{5}{6}) + (60)(\frac{3}{5}) = 50 + 36 = 86$

(d) $(13)(29) = 13(20 + 9) = 260 + 117 = 377$

(e) $(\frac{2}{9} + \frac{5}{6})72 = (\frac{2}{9})(72) + (\frac{5}{6})(72) = 16 + 60 = 76$

(f) $(1.4)(43) + (1.6)(43) = (1.4 + 1.6)(43) = (3)(43) = 129$

5. $3(2x + y) + 5x = (6x + 3y) + 5x$ distributive property

$= (3y + 6x) + 5x$ commutative property of addition

$= 3y + (6x + 5x)$ associative property of addition

$= 3y + (6 + 5)x$ distributive property

$= 3y + 11x$

6. (a) $6x + 3x = (6 + 3)x = 9x$

(b) $41a + 37b + 82a + 14b = (41a + 82a) + (37b + 14b)$

$= (41 + 82)a + (37 + 14)b$

$= 123a + 51b$

(c) $.3x + 1.4y + 7.1x + 1.1z + 2.3y =$

$(.3x + 7.1x) + (1.4y + 2.3y) + 1.1z$

$= (.3 + 7.1)x + (1.4 + 2.3)y + 1.1z$

$= 7.4x + 3.7y + 1.1z$

(d) $\frac{5}{4} + \frac{2}{7}x + \frac{16}{5}y + \frac{5}{7}x + \frac{4}{5}y =$

$$\frac{5}{4} + \frac{2}{7}x + \frac{5}{7}x + \frac{16}{5}y + \frac{4}{5}y$$
$$= \frac{5}{4} + (\frac{2}{7}x + \frac{5}{7}x) + (\frac{16}{5}y + \frac{4}{5}y)$$
$$= \frac{5}{4} + (\frac{2}{7} + \frac{5}{7})x + (\frac{16}{5} + \frac{4}{5})y$$
$$= \frac{5}{4} + \frac{7}{7}x + \frac{20}{5}y$$
$$= \frac{5}{4} + 1x + 4y$$
$$= \frac{5}{4} + x + 4y$$

7. (a) $\{0\}$
   (b) $\{1\}$
   (c) $\{1\}$
   (d) $\emptyset$
   (e) $\emptyset$
   (f) the set of all numbers

8. (a) $5(x + y)$
   (b) $u(4) + (2v)(4)$
   (c) $(3a)(2) + (3a)(4b)$
   (d) $(2 + x)(a + 2)$
   (e) $ab + 2b + a + 2$
   (f) $x^2 + yx + x + y$

Chapter 5

OPEN SENTENCES AND WORD SENTENCES

The purpose of this chapter is to help develop some ability in writing open sentences for word problems. We work first just with phrases. We do some inventing of English phrases to fit open phrases at the start to try to help give a more complete picture of what this translation back and forth is like. Then we translate back and forth sentences involving both statements of equality and inequality.

In order to concentrate on the translation process, we prefer at present not to become involved in finding truth sets of the open sentences. We nevertheless have asked questions in the word problems. This seems necessary in order to point clearly to a variable, to make the experience more closely related to the problem solving to which this translation process will be applied, and to bring forth the sentence or sentences we are really wanting the student to think of. Thus in the first example in the exposition of Section 5-4, if instead of saying, "How long should each piece be?" we said, "Write an open sentence about the lengths of the pieces," the student might well answer, "If one piece is n inches long, then $n < 44$," or he might even answer, "$n > 0$." These are true enough sentences, but they miss the experience we want the students to have.

Some students may feel the urge to go on and "find the answer". In that case you should let them try, but don't let "finding the answer" become a distraction at this point. Tell the students that we will be developing more efficient methods of finding truth sets of sentences later on, but for the present we shall go no further than writing the open sentence.

In a few of the problems in this course there is superfluous information which is not necessary for doing the problem. This is intentional. We hope that occasional experience with such irrelevant material will help make the student more aware of information which is relevant.

An attempt is made throughout the chapter to bring out the point that, in trying to solve a problem about physical entities, one must first set up a mathematical model. Having made the mathematical abstractions corresponding to the physical measures and their relationships, one can then write one or more open sentences, and direct his attention to finding a solution to this mathematical problem. Once such a solution is obtained, it can be related once again to the original physical problem.

5-1. Open Phrases to Word Phrases.

In translating from open phrases to word phrases -- you may prefer to say "English" phrases -- many word phrases are possible. Encourage the students to use their imaginations and bring in as great a variety of translations as possible. It is clear that the broader their experience in this type of translation, the broader will be the base from which they start the reverse process, translation from word phrases to open phrases in the next section. Thus, if supervised study time is available, it may be advisable for the teacher to work with the student as he begins Problem Set 5-1, so the student may be helped to think of a variety of word translations for the given open phrases. If the student says that he cannot think of any different translations, the teacher can ask him (as was done in the text) to respond to the question, "Number of what?" and almost any answer to this question is a substantial beginning for a translation.

In the last example of the text in Section 5-1, it may be necessary for the teacher to work with particular care with the class on the phrase "number of points Bill made if he made 3 more than twice as many as Jim." It seems impossible to simplify this language further, and yet this is typical of a kind of wording that often bewilders a slower student. The teacher should stress that $2x + 3$ is a number.

Possible translations of "-" include "less than," "the difference of," "shorter than," etc. You may have to warn your students that, since subtraction is not commutative, they must watch which number comes first in using "less than."

You will sooner or later find a student who is confused about the difference between "greater than" or "more than" which calls for "+", and "is greater than" or "is more than" which calls for ">". Be prepared to make this distinction clear. Thus, "This turkey weighs five pounds more than that one" could call for the phrase "t + 5", while "This turkey weighs more than twenty pounds" could call for the sentence "$p > 20$".

Answers to Oral Exercises 5-1; page 155:

This exercise is intended to provide experience in translating open phrases when the translation of the variable is given. Pay particular attention to the translations of "$\frac{t}{2}$" since this is a new notion not discussed in the reading. Since the multiplicative inverse and the definition $\frac{a}{b} = a \cdot \frac{1}{b}$ lie well ahead in the text, it will probably be necessary simply to make plausible to the student that $\frac{t}{2} = \frac{1}{2}t$, relying on some examples to do this.

1. (a) One more than the number of quarts of berries that can be picked in one hour
    Two less than the number of quarts of berries that can be picked in one hour
    Twice the number of quarts of berries that can be picked in one hour
    Three more than twice the number of quarts of berries that can be picked in one hour
    One half the number of quarts of berries that can be picked in one hour
   (b) One more than the number of records you can buy for $3
    Two less than the number of records you can buy for $3
    Twice the number of records you can buy for $3
    Three more than twice the number of records you can buy for $3
    Half as many records as you can buy for $3
   (c) One more than the number of feet in the diameter of a given circle
    Two less than the number of feet in the diameter of a given circle
    Twice the number of feet in the diameter of a given circle
    Three more than twice the number of feet in the diameter of a given circle
    One half the number of feet in the diameter of a given circle

Answers to Problem Set 5-1; page 155:

The translations given below are, of course, suggestions only. Encourage students to use different translations. Perhaps you will want to handle these problems as oral exercises. It is advisable that the student should write out the translations for some of the problems but not to the point where it becomes tedious. In Problem 12 the phrase should be translated as it stands. $8x$ is a different phrase from $x + 7x$.

Be sure that the student's response, oral or written, shows that he is aware that the variable represents a number. In this sort of problem, for example, the variable w is not "width" but "the number of feet in the width," x is not "books" but "the number of books Mary has," b is not "the boy" but "the number of years in the boy's age". Notice also that a clear, correct, and smoothly flowing way to say the last phrase is "the boy is b years old".

1. If n is the number of books George read in July, then the phrase is "7 more than the number of books George read in July".
2. If n is the number of pennies Mary had when she went to the store, then the phrase is "the number of pennies Mary has left after she spends 7 of them for candy".
3. If x is the number of inches in Tom's height on his eighth birthday, then the phrase is "the number of inches in Tom's height on his ninth birthday if he grows 2 inches during the year".
4. If x is the number of people that a certain bus can hold, then the phrase is "the number of people in the bus if there are two empty places".
5. If n is the number of couples attending a dance, then the phrase is "the number of people at the dance".
6. If n is the number of miles from A to B, then the phrase is "one more than the number of miles from A to B and back".
7. If n is the number of hats Linda has, then the phrase is "the number of hats Joyce has if she has one less than twice the number Linda has".

8. If n is the number of people in a certain city, then the phrase is "the number of people owning cars if one third of the number of people in that city own cars". The teacher might mention the restriction on the domain of n.

9. If n is the number of oranges on the table, then the phrase is "the number of oranges Tom puts in his basket if his mother first puts another orange on the table and Tom then takes one-third of the oranges and puts them into his basket".

10. If r is a certain number which Harry chooses, then the phrase is "the number Harry gets if he doubles the number he chose and then adds 5 to the result".

11. If r is the number of points made by Frank in his first game, then $2r - 5$ is the number made by Joe if he scores five less than twice as many as Frank.

12. If x is a certain number, then $x + 7x$ is the sum of that number and one seven times as great.

13. If t is the number of students in Mr. White's class, then $\frac{t}{2} + 3$ is the number in Miss Brown's class when her class has three students more than half as many as Mr. White's. Again there is a restriction on the domain of t.

14. If r is the number of dollars Mary has in her purse, then $3r + 1$ is the number of dollars Bill has when he has one dollar more than three times as much as Mary.

15. If t is the number of miles covered by the Jones family on the first of their summer trips, $\frac{2t - 1}{3}$ is the number of miles covered in one third of the second trip if it is to be one mile less than twice the length of the first trip.

5-2. Word Phrases to Open Phrases.

Great care is taken throughout the chapter to point out that a variable represents a number. We have seen that no matter what physical problem we may be concerned with, when we make a mathematical translation we are talking about numbers.

On this point it may seem that, in the example involving line segments given in Section 5-2 of the text, we violate our insistence upon the fact that t is a number. Care should be taken to emphasize in this example that t is indeed a number. The phrasing in the problem, however, is of a kind that the students are going to see. They might as well get used to it and understand that even though we talk this way we are using a variable to represent a number, not as a line segment.

Some of the problems in this chapter may involve more than one variable or may suggest the use of more than one variable. This should be allowed to happen casually. In the case of open sentences you may have opportunity to show the possibility of a compound sentence. It is too early to be able to show the necessity of a compound sentence for a unique solution. Since we are not at present looking for answers it will not be necessary to worry yet about how we will find the truth set. Nevertheless, the student should be encouraged to use one variable only whenever he is able to, so that, for example, consecutive whole numbers would be represented by x, $x + 1$, and $x + 2$, rather than by x, y and z. If the examples in the text have been at all effective and if the translations of the previous sections were done satisfactorily, then it seems likely that the use of more than one variable will be, for most students, a sort of last resort measure. In many of these cases the teacher can aid the student in thinking through the problem again so as to permit the student in effect to redefine one variable in terms of another.

Answers to Oral Exercises 5-2; pages 157-158:

Help the student to notice that when the variable is given in the problem, it is not necessary for him to tell about it, but if the problem does not give the variable, it is the student's responsibility to choose a letter and tell what it represents. Exercises 8 through 13 require the student to choose the variable. Encourage the use of different letters of the alphabet. By this means it is hoped that students will realize that the meaning or definition of the symbol is the important consideration rather than the choice of the symbol to be used.

For many of the following problems there are implied restrictions on the domain of the variable. While we ordinarily let such restrictions remain implied because they seem quite obvious, there would be some value in occasionally discussing with the students what restrictions actually exist. For instance, in Exercises 8 and 9, the domain of the variable is the set of multiples of $\frac{1}{100}$; that is, when the variable represents a number of dollars, the domain cannot include numbers like $\frac{3}{7}$. In Exercises 17 and 18, the domain is the set of whole numbers; in Exercise 19, the set of multiples of $\frac{1}{2}$; in Exercise 20, the set of multiples of $\frac{1}{3}$.

Of course, at this point in the course we are restricting ourselves to the numbers of arithmetic, but most of the problems of this chapter by their nature give only non-negative numbers in the domain anyway.

1. $k + 7$
2. $25t$; $100n$
3. $n + 5$
4. $n - 5$
5. $5n$
6. $n + 5$
7. $14x$
8. If q is the number of dollars in the bank, then the phrase is $q + 7$.
9. If s is the number of dollars in the bank, then the phrase is $s - 7$.
10. If Sam is b years old, then the phrase is $b + 4$.
11. If Sam's age is m years, then the phrase is $m - 3$.
12. If Sam is q years old, then the phrase is $2q$.
13. If Sam is c years old, then the phrase is $\frac{1}{2}c$.
14. $12x$
15. $3y$
16. $36t$
17. $5k$
18. $10d$
19. $50y$
20. $60n$

Answers to Problem Set 5-2; pages 158- 160:

The teacher should be prepared to teach or reteach the ideas of perimeter and area which are used in these problems.

pages 158-159: 5-2

1. If n is the number of dollars Fred has,
(or, "if Fred has n dollars")
$3n + 7$

2. If n is the number of dollars Ann has,
$3n - 7$

3. If w is the number of inches in the width of the rectangle,
(or, "if the rectangle is w inches wide")
$2w$

4. If n is the number,
$n + 2n$ (Leave it in this form, $3n$ is not a translation of the phrase.)

5. If c is the counting number,
$c + (c + 1) + (c + 2)$ (leave it that way)

6. If q is the even number,
$q + (q + 2)$

7. If n is "some" number,
$(n + 3)2$ or $2(n + 3)$

8. If n is "some" number,
$2n + 3$

9. If the rectangle is n inches wide, $n(n + 10)$ (You may have to remind them how to find area. Be certain that n is the number of inches in the width, not just n is the width.)

10. If w is the number of inches in the width of the rectangle,
$w + (w + 10) + w + (w + 10)$
(Here they may think 2 times the number of inches in the width and 2 times the number of inches in the length, so accept $2w + 2(w + 10)$.)

11. If s is the number of units in the side of the square,
$s + s + s + s$
Or $4s$ if they arrive at this by thinking of 4 times the number of units in the side

12. $100t + 25(t + 2)$ is the number of cents.

13. $10d + 25(d + 5)$ is the number of cents.

14. $60n + 70(n + \frac{1}{2})$ is the cost in cents.

pages 159-162: 5-2 and 5-3

15. $.1n$ is the number of ounces of salt.

16. $100k + 25(k - 2) + 5(2k)$ is the number of cents.

17. $36.9g + 30.9(g + 2)$ is the number of cents.
$.369g + .309(g + 2)$ is the number of dollars.

18. $50q + 32.5(7q)$ is the number of cents.
$.50q + .325(7q)$ is the number of dollars.

19. $c + 10(2c) + 5\cdot\frac{1}{2}(c + 5) + 100(c - 7)$ is the number of cents.

20. $10y + 5(y + 2)$ is the cost in cents.

21. If x is the number of candy bars, then $(\frac{x}{3}) \cdot 10 + (\frac{x + 4}{4})50$ is the cost in cents.

22. If t is the number of dimes, then $10t + 25(t + 2) + 5(\frac{1}{2}t)$ is the total value in cents.

23. If r is the number of pounds of solution, then $.25r$ is the number of pounds of salt.

24. If s is the number of units in the length of the rectangle, then $2(s + \frac{3}{4}s)$ is the number of units in the perimeter.

25. If q is the number of units of the width of a box, then $(q)(2q)(\frac{3}{8}q)$ is the number of cubic units of volume.

Answers to Oral Exercises 5-3; pages 161-162:

1. If t is the number of students in John's class, the translation is:

   Ninety is eight more than the number of students in John's class.

2. If y is the number of cents John receives each time he mows Mrs. Jones' lawn, the translation is:

   John receives $1.05 for mowing Mrs. Jones' lawn five times.

3. If x is the total number of students in school, the translation is:

   When the total number of students in school is divided among ten gym classes, the average number in each class is 47.

page 162: 5-3

4. If n is the number of feet in the length of a piece of board, the translation is:

   A second piece, which is 62 feet long, is eight feet more than twice the length of the first.

5. If x is the number of votes Joe receives, the translation is:

   The number of votes Joe receives, decreased by five, equals 12, the number of votes received by John.

6. If n is the number of units in the length of one piece of paper, the translation is:

   The length of paper needed to make two posters is 30 inches, if one poster is one inch more than twice the length of the other.

7. If r is the number of units in the length of a rectangle, the translation is:

   The area of a rectangle is 18 square units, if the width is three units less than the length.

8. If r is the number of units in the length of a sheet of construction paper, the translation is:

   A sheet of construction paper does not have an area of 18 square inches, if its width is 3 inches shorter than its length.

9. If t is the number of yards gained in the first play of a football game, the translation is:

   The team gained twenty yards in two plays. In the second play the team gained one yard less than 3 times the number of yards gained in the first play.

10. If t is the number of dollars Mike has, the translation is:

    The number of dollars Robert has is one dollar less than three times the number Mike has. John, who has 20 dollars, does not have the same number of dollars as Robert and Mike have together.

Answers to Problem Set 5-3; page 162:

We give suggested translations for the open sentences; assigning them to the pupils will produce a great variety of translations. One way of testing the correctness of the student translations might be to distribute them about the class and have the pupils try translating them back into open sentences. This would also serve to give the pupils a start on the work of the following section by having them first translate pupil-made problems into open sentences.

1. Let n be the number of books in Bill's desk. Five times the number of books in Bill's desk is 25.
2. Let y be the number of years in Harry's age now. Five years from now Harry will be twenty years old.
3. Let t be the number of inches in the length of the board. After five inches is sawed off a board the remaining piece is 20 inches long.
4. Let t be the number of dollars in the total amount. Each of five persons received 20 dollars when the money was divided.
5. Let n be the number of dollars Frank has. John has three dollars. Two times the number of dollars Frank has plus what John has is 47 dollars.
6. Let n be the number of firecrackers Frank bought. John bought twice as many firecrackers as Frank did. After he used 3 he had 47 left.
7. Let x be the number of inches in one side of the square. The perimeter of a square is 90 inches.
8. Let n be the number of dresses Jean has. Mary had 4 times as many dresses as Jean. Alice had 7 times as many as Jean. Together Alice and Mary had 44.
9. Let k be the number of hours Harry and Bill rode. Harry and Bill rode their bikes for the same length of time. Harry traveled 5 miles per hour, Bill traveled 12 miles per hour. They traveled in opposite directions and were 51 miles apart at the end of this period of time.
10. Let n be the number of feet in the width of the rectangle. The length of a rectangle is twice the width. Its area is 300 square feet.

11. Let n be the number of feet in the width of the rectangle. The length of a rectangle is two feet more than the width. Its area is 300 square feet.
12. Let w be the number of feet in the longer side of the rectangle. One side of a rectangle is four feet less than the other. Its area is 16 square feet.
*13. Let x be the number of dollars John has. Jim has one dollar more than three times the number John has. Together they have 46 dollars.
*14. Let y be the number of blocks Bill walked. John walked 3 blocks after walking twice as far as Bill. Tom walked 3 blocks after walking the same distance as Bill. John and Tom walked a total of 30 blocks.

5-4. Word Sentences to Open Sentences.

In this lesson, we turn our attention to verbal problems. You will notice that a question is asked in each of the problems. Earlier in the commentary it was pointed out that the question serves to help the student ferret out the number he is interested in and to make the most fruitful translation.

The "guessing" method employed in the examples of this lesson is usually an effective one for students who are troubled by the abstraction of switching from a word problem to an open sentence. You may want to make even greater use of this approach than indicated in the text. For many students, this guessing technique may remain the best way to make translations independently.

The short exposition on page 165 concerning "is less than" and "is 5 less than" results from past experience in which many students tend to see these phrases as saying essentially the same thing. Thus, such meaningless translations of "5 is 4 less than 9" as "$5 = 4 < 9$" have arisen. Hence the warning to the student at the end of this section.

Answers to Oral Exercises 5-4; pages 166-167:

The emphasis in Exercises 3-19 is on the translation to sentences. Exercises 8, 9, and 11 through 22 involve variables.

pages 166-167: 5-4

It is not essential that the truth set be found, but if the students want to do so, permit them to have this fun.

1. (a) n (any variable may be chosen)
   (b) $n - 8$
   (c) $n(n - 8)$
   (d) $n(n - 8) = 180$

2. (a) r represents the number of inches of length of the short piece.
   (b) $r + 3$
   (c) $r + (r + 3)$
   (d) $r + (r + 3) = 39$

3. $30 = 17 + 13$
4. $14 = 17 - 3$
5. $14 < 10$
6. $14 = 10 - 7$ This is a false sentence, but it is a sentence.
7. $42 = 32 + 10$
8. $42 = x + 10$
9. $42 < x + 7$
10. $12 = 21 - 9$
11. $12 = x - 9$
12. $x = 12 + 4$
13. $y = 32 - 12$
14. $y > 32 - 12$
15. $2m = m + 3$
16. $s = 2s - 5$
17. $s < 21 + 5$
18. $15 = 3x - 2x$
19. $5 = 4y + y$
20. If n is the number, $n = 2n - 3$.
21. If r is the number, $r < 5r + 3$.
22. If q is the number, $3q > 2q + 5$.

Answers to Problem Set 5-4; pages 167 - 170:

Most of these problems do not have integral solutions. This is to prevent the student from guessing the right answer before he has written the open sentence. The emphasis is on the open sentence, not the truth set.

page 167: 5-4

Several important points might be mentioned again at this time in an effort to anticipate and forestall translation errors by the students:

1. The question asked in the problem is the most effective guide to the student in the definition of the variable. (Note that the variable need not always be the number which is the answer to the problem, though this will often be the case.)

2. Any other numbers needed in the problems should be stated in terms of the one named by the variable. Thus we say, "If the shorter piece is x inches long, the longer piece is $(x + 3)$ inches long." Of course some situations may naturally lend themselves to the use of two variables. As we have said before, there is no objection in this chapter to including an occasional example of this sort.

3. There should be a direct translation into an open sentence. Thus in Problem 1 of this Problem Set, while we could change the sentence to $3x = 80$, such a sentence is not a direct translation of the problem. It does not really tell the story. A good test of a direct translation is to see whether, with the description of the variable, the sentence can be translated readily back into the original problem.

The form in which the student is to write these problems is suggested in the examples in the text. Some freedom of form is desirable, of course, but certainly a clear definition of the variable should appear along with the sentence. Frequently the student will find it helpful to write out phrases, especially the more complicated ones, in terms of the variable, before writing the sentence. Thus a typical example might have this appearance:

1. If n is the number,
   then $2n$ is twice the number,
   and $n + 2n = 80$.

Other problems will occasionally be written out in this manner in the answers below; in most cases, however, only the sentence is written, since the form is similar in all problems.

2. $3x - x = 15$
3. $106 = x(x + 5)$
4. $82 = x(x - 6)$

5. $58 = x + 2x$
6. $10 + 20 + x = 60$
7. $10 + 2x + x = 60$
8. $x + 2x + 3x = 40$
9. $x + x + 13 + 13 = 66$ or $2x + 26 = 66$
10. $w + w + (w + 5) + (w + 5) = 60$ or $2w + 2(w + 5) = 60$
11. $(x + 7) + (x + 7) + x + x = 50$
12. $(x - 6) + (x - 6) + x + x = 72\frac{1}{2}$
13. $x(2x) = 98$
14. $x + (x + 30) = 516$
15. $5000 + 2x + x = 10{,}500$
16. $5x = 175$
17. $5x + 3(10) = 175$
18. $5x + 2x(10) = 175$
19. $6(5) + x(10) = 60$
20. $x(10) + 2x(5) = 60$
21. $x(25) + (3x - 1)10 = 155$
22. $2(25) + 2x(10) + x(5) = 200$
23. $x(10) + (x + 1)25 + (2x + 1)5 = 165$
24. $(x + 2) + x = 64$
25. $21(4) + 3(3) + x(2) + 3x(1) = 153$
26. $(x - 3) + (3x - 3) = 22$

*27. Let n be the number of nickels, and d the number of dimes. Then $d + 2$ is the number of quarters. We obtain the open sentence

$$5n + 10d + 25(d + 2) = 325$$

The fact that n and d can be only positive integers makes it possible to determine seven solutions. The possible values for d are 1, 2, 3, 4, 5, 6, and 7, and the corresponding values for n are 48, 41, 34, 27, 20, 13 and 6.

*28. $(x + 3) + (2x + 3) = 30$

*29. $x + 3 = 3x - 3$, if the table is x feet wide

*30. $5(x + 20) = 5x + 100$, if the speed of the freight is x miles per hour. Notice that all values of x are truth numbers of the open sentence.

*31. Let x be the number of quarters; then $3x$ is the number of dimes and $2x$ the number of nickels. Let y be the number of cents John has. Then

$$y = 25x + 10(3x) + 5(2x).$$

The solution set would consist of an infinite set of pairs of positive integers of the form $(x, 65x)$.

5-5. Other Translations.

Here we extend the student's experience with translating to sentences involving inequalities. The term "inequality" as well as the word "equation" is not introduced in the text until Chapter 9. The exposition in this section of the text parallels the earlier presentation of sentence translation in Sections 5-3 and 5-4.

While we are not at the moment concerned with finding the truth sets of sentences, it is likely that the student will be at least occasionally interested in discovering the "answers" to the problems for which he has written sentences. Thus it is almost certain that it will be noticed and pointed out that inequalities frequently have many numbers in their truth sets, instead of just one, as was often the case with equations. Here the idea of the open sentence as a "sorter" of the domain of the variable can be re-emphasized. All elements of the domain which make the sentence true are possible "answers" to our word problem, and those elements of the domain which make the sentence false cannot be "answers" to the problem. Though we lack a definitive single "answer," we have a clearly defined set of "answers," i.e., the truth set of the sentence.

Answers to Oral Exercises 5-5a; page 171:

(These are possible translations.) The teacher may want to omit some of the latter exercises of this set, particularly if it seems that prolonged background discussion is needed regarding the geometry upon which the translations would be based.

1. If a is the number of boys in class, the number of boys in class is less than 3.
2. If a is the number of dollars in Joe's pocket, the number of dollars is greater than 3.

page 171: 5-5

3. If n is the number of books needed, the number of books increased by one is greater than 17.
4. If n is the number of points made by Harry, one point more than the number is less than 17.
5. If t is the number of pencils, four more than three times the number is less than 12.
6. If John is x years old, John's age is greater than 10 and less than 15.
7. If m is the number of hours required to do a job, the time required is at least 3 hours and no more than 12 hours.
8. If n is the number of yards gained on the first play, and the second play gained five yards more than the first, the sum of the yardage gained on the two plays is greater than 35 yards.
9. If a is the score earned by Mary, b is the score earned by Jane, and c is the score earned by Mike, Mary's score is greater than Jane's and the sum of Mary's and Jane's scores is greater than Mike's.
10. If a bag of change contains n nickels, two more dimes than nickels, and one less quarter than nickels, the sum of money in the bag is greater than 4 dollars.
11. If a is the number of units in the base of a triangle and if the height is two units more, the area of the triangle is greater than 20 square units.
12. If ℓ is the number of units in the width of a rectangle and if the length is one unit more, the area is not more than 37 square units.
13. If the radius of a circle is increased by one, the area of the new circle is at least 40 square units.
14. If the height of a cylinder is two units greater than the radius, a, of the base, the volume of the cylinder is less than 17 cubic units.
15. A cylinder with height 4 has a radius shorter than the height of a box. The box has a base with area 6. The volume of the cylinder is at least as great as the volume of the box.

16. If x pounds of salt are used to make a 10% solution, the amount of salt in the solution is 9 pounds.
17. If x pounds of candy selling for \$.30 a pound is mixed with some weighing two pounds more and selling at \$.40 a pound, the mixture is worth \$3.40.
18. If the height of a triangle is one unit more than the base, b, the area is no more than 15 square units.
19. If the width of a rectangle is two inches less than the length, the perimeter is less than 19.
20. If one side of a triangle is twice the first side, a, and the third side is one less than three times the first, the perimeter is greater than $12\frac{1}{2}$.

Answers to Problem Set 5-5a; page 172:

The following are suggested translations. Encourage a variety of translations.

1. If t is the number of boys in the club, the number of boys in the club is less than 6.
2. If t is the price of a sweater in dollars, the price of the sweater is greater than 6 dollars.
3. If y is the number of students in the class, the class will have less than 60 students when 15 more join.
4. If y is Jimmy's score on a test, he will have a score of more than 60 if he gets 15 points bonus.
5. If y is the number of cars on the parking lot, a lot 10 times as big could hold more than 80 cars.
6. If r is the price of a stamp in pennies, 25 stamps would cost less than 2 dollars.
7. If x is the length of a section of fence in feet, two sections of fence plus a gate that is 5 feet long will cover more than 50 feet.

In the early part of our work with translation we have been trying to emphasize the idea that the variable represents a number by being reasonably precise in the language. Thus we have been saying, "the number of dollars in the price of the sweater" or the "number of inches in the length of a rectangle". As we go on, we will allow ourselves to become more relaxed in order to speak more fluently. Thus, we may say, "x is the length of a section of fence in feet" when there is no doubt that we mean "x is the number of feet in the length of a section of fence".

8. If x is the weight in pounds of a sack of flour, two sacks of flour plus 5 pounds of sugar weigh less than 50 pounds.
9. If a is the number of years in Mary's age, if Jane is twice as old as Mary, and if Sally is three times as old as Mary, the sum of their ages is more than 48.

*10. Same as above...sum of their ages is greater than or equal to 48.

Answers to Oral Exercises 5-5b: pages 174-175:

1. If x is the number of dollars John has, $x > 50$.
2. If y is the number of students living in the city, $y < 150$.
3. If r is the height of the plane in feet, $r \leq 30,000$.
4. If s is the height of the plane in feet, $s \leq 30,000$ and $s > 5280$.
5. If q is John's weight in lbs., $q + (q + 10) > 220$.
6. If b is the number of brothers Jane has and c is the number of brothers Mary has, $c > b$.

Answers to Problem Set 5-5b; pages 175-176:

1. If x is the number of dollars Tom has, $x > 200$.
2. If y is the number of people that went to the park, $y < 100$.
3. If t is the number, $4t + 9t > 100$.
4. If r is the number, $7r \geq 45$.
5. If n is the number, $8n - 3n \leq 10$.
6. If h is the altitude in feet at Denver, $h > 5000$.
7. If m is the number of people who live in Mexico, $180,000,000 > 2m$.
8. If r is the number of years in Norma's age, $r + (r + 5) < 23$.

10. If h is the number of hours on the job, $h > 2$ and $h \leq 4$.
11. If k is the number killed, $k \geq 250$ and $k \leq 300$.
12. If c is the speed of the current, $c + 12 < 30$.
13. If m is the number of minutes of advertising, $m \geq 3$ and $m < 7$.
14. If x is the length of a side of the square, $x + x + x + x = (x + 5) + (x + 5) + (x + 5)$
15. If r is the number of students who remained and n is the number of students enrolled, then $r < \frac{1}{2} n - 152$. This problem is similar to Exercise 6 of Oral Exercises 5-5b, in that a sentence for it must be expressed in terms of two variables. There will doubtless be considerable discussion of both exercises.

Summary.

The first part of the summary is a parting effort to strengthen the idea of translation back and forth between a physical situation and a mathematical (or numerical) one.

The latter part of the summary reviews, by means of examples, the kinds of translations that have come up in this chapter.

Answers to Review Problem Set; pages 178-183:

1. Let t be the number of marbles in one jar.
   (a) The number of marbles in 2 jars
   (b) 3 more than the number of marbles in one jar
   (c) The number of marbles in 3 jars, each holding as many as the first, after two marbles have been removed
   (d) The number of marbles in one jar after one has been removed
   (e) After one marble is removed from one jar, 5 marbles are left in the jar.
   (f) If we take out one half of the marbles in one jar, we will take out less than 4 marbles.
   (g) If we count the marbles in two jars the number of marbles is greater than 6.
   (h) There are at least 6 marbles in one jar.

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2. (a) $7w$, w is the number of weeks.
   (b) $x(2x)$, x is the number.
   (c) $3x + 5$, x is the number of students.
   (d) $7(x - 5)$, x is the number from which 5 is to be subtracted.
   (e) $\frac{1}{2}(x(2x))$, x is the number of units in the length of the shorter side of the rectangle.
   (f) $5x + 10(2x)$, x is the number of nickels.
   (g) $1.40\,x + .30(x + 1)$, x is the number of pounds of chocolates.
   (h) $\frac{1}{2}(x)(x + 3\frac{1}{2})$, x is the number of units in the base.
   (i) $\pi\, x^2(x + 5\frac{3}{4})$, x is the number of inches in the radius of the base.
   (j) $\frac{(x + 2\frac{1}{3})\ 8.9}{3.4}$, x is the number.
   (k) $.20\,x$, x is the number of gallons of salt solution.
   (l) $2x - 4$, x years is Mary's age now.
   (m) $25x + 32(x + 2)$, x is the number of loaves of bread.
   (n) $25x + 10(x + 3) + 5(x - 2)$, x is the number of quarters.
   (o) $\frac{1}{2}(x)(\frac{1}{2}\,x - 2)$, x is the length of the base.
   (p) $\frac{\frac{7x}{3}}{2} + 2x$, x is the original number.
   (q) $\frac{1}{2}(x + (x + 3\frac{1}{2}))(x - 1)$, x is the number of inches in the length of the shorter base.
   (r) $x + 2x + 2(2x)$ x is the number of units in the length of the shortest side.

3. (a) $x + 3x = 45$, x is the number.
   (b) $x + (x + 1) = 45$, x is the first number.
   (c) Insufficient information
   (d) $x + 4 = 16$, x years is Mary's sister's age.
   (e) $x > 14{,}000$, x feet is the height of Pike's Peak.
   (f) $x + (x + 2) = 75$, x is the first odd number. This problem has no solution since the sum of two odd numbers is even, but we can still write the open sentence.
   (g) $3x \geq x + 26$, x is the number of students in the class.

(h) $x + (2x + 20) + 70 = 180$, x is the number of degrees in the measure of the smallest angle. Here the variable x does not represent the number of degrees in the measure of the largest angle which appeared in the question.

(i) $x + (x + 48) = 1124$, x is the smaller even number.

(j) $x = 18 + 22$, x is the sum of the numbers of years in the ages of each 6 years from now.

(k) $(x + 1)^2 - x^2 = 27$, x units is the side of the smaller square.

(l) $8x = 12(5 - x)$, x hours is the time spent riding into the country.

(m) $2(7x + x) = 150$, x inches is the width of the rectangle.

(n) $\frac{2}{3}x + 32 = 38\frac{1}{2}$, x is the number.

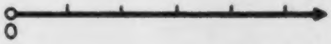
(o) $.10x = .025(x + 20)$, x is the number of pounds of the original solution.

(p) $x - .20x = 29.95$, x dollars is the original price.

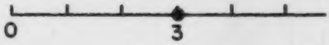
(q) $62\frac{1}{2}x - 39.7x = 125\frac{1}{2}$, x hours is the required time.

(r) $x + (x + 4) + ((x + 4) + 6) = 50$, x is the number of pennies.

4. (a) all numbers greater than zero

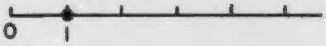


(b) $\{3\}$

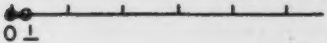


(c) $\{\frac{3}{2}\}$

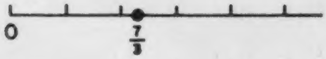
(d) $\{1\}$



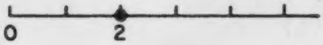
(e) all numbers less than $\frac{1}{5}$



(f) $\{\frac{7}{3}\}$

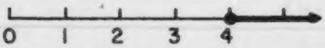


(g) $\{2\}$

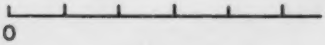


(h) $\emptyset$

(i) all numbers equal to or greater than 4



(j) $\emptyset$



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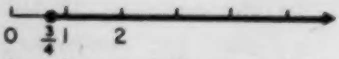
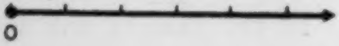
(k) all numbers greater than or equal to $\frac{3}{4}$



(l) all numbers



5. (a) $a(2b + 3a)$
(b) $a(3 + 1)$
(c) $x(2b + 1)$
(d) $ax(\frac{4}{3} + \frac{1}{3})$ or $a(x(\frac{4}{3}) + x(\frac{1}{3}))$ or $x(a(\frac{4}{3}) + a(\frac{1}{3}))$ or $\frac{1}{3}ax(4 + 1)$
(e) $(\frac{4}{3} + \frac{1}{3})ax$ or other answers as in (d) above.
(f) $2(\frac{7}{8}a + \frac{3}{4}x)$
(g) $\frac{5}{2}m(1 + \frac{9}{10})$
(h) $x(2a + 1)$
(i) $(7 + 6)2$
(j) $(a + b)(x + y)$

6. (a) $3x^2 + x$
(b) $6a^2 + 4a$
(c) $a^2b^2 + 7a^2b$
(d) $10a^2 + 10ab$
(e) $12c^2d + 12cd$
(f) $\frac{8}{9} m + \frac{4}{3} m^2$
(g) $\frac{4}{10} x^2 + \frac{12}{40} x$
(h) $8.25x + 3.96$
(i) $ac + ad + bc + bd$
(j) $10 + 8$

7. (a) No. 1 + 3 = 4. 4 is not an element of the set.
(b) Yes. 2 + 2 = 4, 2 + 4 = 6, 2 + 6 = 8, etc.
(c) No. 1 + 5 = 6. 6 is not an element of the set.
(d) No. 30 + 2 = 32. 32 is not an element of the set.
(e) No. 30 + 5 = 35. 35 is not an element of the set.
(f) Yes. 0 + 0 = 0
(g) No. 1 + 1 = 2. 2 is not an element of the set.
(h) No. 1 + 3 = 4. 4 is not an element of the set.
(i) No. 5 + 10 = 15. 15 is not an element of the set.
(j) No. 10 + 10 = 20. 20 is not an element of the set.

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8. (a) 3
   (b) 7
   (c) $\frac{23}{12}$
   (d) 4
   (e) 56
   (f) $\frac{17}{6}$
   (g) $\frac{5}{2}$

9. (a) $\frac{29}{12}$
   (b) $\frac{25}{24}$
   (c) $\frac{7}{24}$
   (d) $\frac{15}{4}$
   (e) 6
   (f) 9
   (g) 0
   (h) 0
   (i) 0
   (j) $\frac{13}{6}$

Suggested Test Items

1. Translate the following word phrases into open phrases.
   (a) three less than twice the number n
   (b) the product of x and the number which is 7 times x
   (c) the sum of 5 times a number d and a number 4 greater than d
   (d) the number of marbles Jimmy has if he had m marbles and was given 10 more
   (e) the value in cents of n nickels and 4 pennies

2. Translate each of the following into an open phrase or into an open sentence, using a single variable in each. First tell what the variable represents.
   (a) $30 more than Jim's weekly salary
   (b) Tom's weekly salary is more than $30.
   (c) Tom's weekly salary is $30 more than Jim's. Together they earn $140 per week.
   (d) Tom's weekly salary is $30 more than Jim's. Together they earn more than $140 per week.

3. Write a word translation for each of these phrases. Make your word phrase as meaningful as possible.

(a) $2c + 5$ (d) $10m + 25m$

(b) $8n$ (e) $4(a - 3)$

(c) $\ell(\ell - 7)$ (f) $x + 3(x - 2)$

4. Write an open phrase which describes the following statement:
Choose a number and then add 4 to it. Multiply this sum by 3. Subtract 5 from this product.

5. If p is the first of three consecutive odd numbers, then
the second odd number is ___________
the third odd number is ___________
the sum of the first and third numbers is ___________

Complete the following two problems so that each problem corresponds to the given open sentence.

6. Open Sentence: $a + 4a + 25 = 180$
Problem: The perimeter of a triangle is 180 inches.

7. Open Sentence: $5(x + 4) + 10x = 125$
Problem: John found a billfold containing $125.

Write an open sentence or phrase for each of the following:

8. The area A in square feet of a rectangle whose length is x yards and width is y feet.

9. In an orchard containing 2800 trees, the number of trees in each row is 10 less than twice the number of rows. How many rows are there?

10. Bill weighs 10 pounds more than Dave. Find Dave's weight if the combined weight of the two men is 430 pounds.

11. Jack is 3 years older than Ann, and the sum of their ages is less than 27 years. How old is Ann?

12. The number of cents Paul has if he has d dimes and three times as many quarters as dimes.

13. If a boy has 250 yards of chicken fence wire, how long and how wide can he make his chicken yard, if he would like to have the length 25 yards greater than the width?

14. There are five large packages and three small ones. Each large package weighs 4 times as much as each small one, and the eight packages together weigh 34 pounds 8 ounces. What is the weight of each package?

15. Separate \$38 into two parts such that one part is \$19 more than the other.

16. The thickness of a certain number of pages of a book if each page is $\frac{1}{400}$ of an inch thick.

17. The product of a whole number and its successor is 342. What is the number?

18. A father earns twice as much per hour as his son. If the father works for 8 hours and the son for 5 hours, they earn less than \$30. How much does the son earn per hour?

Answers to Suggested Test Items

1. (a) $2n - 3$
   (b) $x(7x)$
   (c) $5d + (d + 4)$
   (d) $m + 10$
   (e) $5n + 4$

2. (a) Let x be Jim's weekly salary in dollars.
   The translation of the phrase: $x + 30$
   (b) If Tom's weekly salary is x dollars, then the translation is: $x > 30$
   (c) If Jim's weekly salary is x dollars, then Tom's weekly salary is $(x + 30)$ dollars.
   The translation: $x + (x + 30) = 140$
   (d) If Jim's weekly salary is x dollars, Tom's weekly salary is $(x + 30)$ dollars.
   The translation: $x + (x + 30) > 140$

3. (Possible translations)
   (a) 5 more than twice the number of pennies Jimmy has
   (b) 8 times as many rainy days as in June
   (c) the area in square feet of a rectangle whose width is 7 feet less than its length
   (d) the total cost in cents of a certain number of ice cream cones at 10¢ each and the same number of sodas at 25¢ each
   (e) the perimeter in inches of a square whose side is 3 inches shorter than the side of a given square
   (f) the perimeter in inches of a quadrilateral three of whose sides are of equal length and the fourth side 2 inches longer

4. $3(x + 4) - 5$

5. $p + 2$
$p + 4$
$p + (p + 4)$ which is $2p + 4$

6. One side is four times as long as another side and the third side is 25 inches in length. Find the length of each side.

7. There were 4 more \$5 bills in the billfold than \$10 bills. How many \$10 bills did John find?

8. $A = 3xy$

9. Let n be the number of rows. The number of trees in each row is $2n - 10$.
$$n(2n - 10) = 2800$$

10. If Dave weighs x pounds,
$$x + (x + 10) = 430.$$

11. If Ann is x years of age,
$$x + (x + 3) < 27$$

12. $10d + 25(3d)$

13. Let the width of the yard be x yards.
$$2x + 2(x + 25) = 250$$

14. Let the weight of each small package be x pounds. Then each large package weighs $4x$ pounds.
$$3x + 5(4x) = 34\frac{1}{2}$$

15. $x + (x + 19) = 38$, where x is the number of dollars in the smaller part.

16. $\frac{1}{400}x$ which can be written $\frac{x}{400}$

17. $n(n + 1) = 342$ where n is the smaller whole number

18. Let m be the number of dollars the son earns per hour. Then the father earns $2m$ dollars per hour.
$$5m + 8(2m) < 30$$

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CHALLENGE PROBLEMS

In the event that you should have an exceptionally interested and eager student we have tried to include a few problems of varying difficulty but usually requiring more perseverance and insight than most problems in the text. We do not recommend these for class discussion or as assigned problems for the entire class. There are, of course, many other resources for challenge problems. We recommend publications of the National Council of Teachers of Mathematics and Dover Publications among others.

Answers to Challenge Problems; pages 184-190:

1. $(8 \times 3) + 2 = 26$
$8 \times (3 + 2) = 40$
$(8 \times 3) - 2 = 22$
$8 \times (3 - 2) = 8$
$8 \times (3 \times 2) = 48$
$(8 \times 3) \times 2 = 48$

$8 + (3 + 2) = 13$
$(8 + 3) + 2 = 13$
$8 + (3 \times 2) = 14$
$(8 + 3) \times 2 = 22$
$8 + (3 - 2) = 9$
$(8 + 3) - 2 = 9$

$8 - (3 \times 2) = 2$
$(8 - 3) \times 2 = 10$
$8 - (3 + 2) = 3$
$(8 - 3) + 2 = 7$
$8 - (3 - 2) = 7$
$(8 - 3) - 2 = 3$

This is an interesting study in arrangements. The 8, 3, and 2 are fixed. The first of the signs may be ×, +, or -, and for each of these the second of the signs may be ×, + or -. Then there are two ways in which parentheses may be inserted, grouping either the first two terms or the last two. After all this is done, it is interesting to note which expressions are names for the same number. For instance:

$$8 - (3 + 2) = (8 - 3) - 2$$
$$8 + (3 + 2) = (8 + 3) + 2$$
$$8 \times (3 \times 2) = (8 \times 3) \times 2$$

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2. $$\begin{aligned}19(13) &= 19(10 + 3)\\ &= 19(10) + 19(3) && \text{distributive property}\\ &= 19(10) + (10 + 9)3\\ &= 19(10) + (10(3) + 9(3)) && \text{distributive property}\\ &= (19(10) + 10(3)) + 9(3) && \text{associative property of addition}\\ &= (19 + 3)10 + 9(3) && \text{commutative property of multiplication and distributive property}\\ &= 220 + 27\\ &= 247\end{aligned}$$

(a)

$$\begin{aligned}15(14) &= (15 + 4)10 + 5(4)\\ &= 190 + 20\\ &= 210\end{aligned}$$

$$\begin{aligned}14(15) &= (14 + 5)10 + 20\\ &= 190 + 20\\ &= 210\end{aligned}$$

(b)

$$\begin{aligned}13(17) &= (13 + 7)10 + 3(7)\\ &= 221\end{aligned}$$

$$\begin{aligned}17(13) &= (17 + 3)10 + 3(7)\\ &= 221\end{aligned}$$

(c)

$$\begin{aligned}11(12) &= (11 + 2)10 + 2\\ &= 132\end{aligned}$$

$$\begin{aligned}12(11) &= (12 + 1)10 + 2\\ &= 132\end{aligned}$$

This is really a trick, although it has an algebraic explanation. It may be that pupils will accept it and use it to simplify mental multiplications. This is good, but not a requirement. In any discussion with a student it should be made clear that the development hinges on the use of 10 as a factor and thus the procedure should be used only for numbers between 10 and 20.

3. $10 + a$ is the first number

$10 + b$ is the second number

$$\begin{aligned}(10 + a)(10 + b) &= 100 + 10a + 10b + ab && \text{by the distributive property}\\ &= 10(10 + a + b) + ab && \text{by the distributive property}\\ &= 10((10 + a) + b) + ab && \text{by the associative property of addition}\end{aligned}$$

$10 + a$ is the first number, and b is the units digit of the second number, so $(10 + a) + b$ is the sum of the first number and the units digit of the second number. $10((10 + a) + b)$ is the result of multiplying the sum of the first number and the units digit of the second number by 10. $10((10 + a) + b) + ab$ is the complete translation of the rule.

4. $$\begin{aligned} 35874 &= 3(10{,}000) + 5(1000) + 8(100) + 7(10) + 4 \\ &= 3(9999 + 1) + 5(999 + 1) + 8(99 + 1) + 7(9 + 1) + 4 \\ &= 3(9999) + 3 + 5(999) + 5 + 8(99) + 8 + 7(9) + 7 + 4 \\ &= (3(1111) + 5(111) + 8(11) + 7(1))9 + (3 + 5 + 8 + 7 + 4) \end{aligned}$$

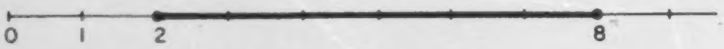
Since $(3 + 5 + 8 + 7 + 4) = 27 = 3(9)$, we see that 35874 is divisible by 9. The general rule to be formulated is "A number is divisible by 9 if the sum of its digits is divisible by 9."

Since we hope to teach pupils to generalize, it would be well to take this opportunity to do just that: Let the thousand's digit of a four digit number be represented by a, the hundred's digit by b, the ten's digit by c, and the unit's digit by d. Then the number is:

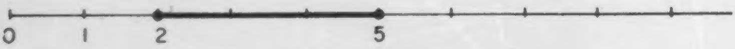
$$\begin{aligned} 1000a + 100b + 10c + d &= (999 + 1)a + (99 + 1)b + (9 + 1)c + d \\ &= 999a + a + 99b + b + 9c + c + d \\ &= 999a + 99b + 9c + a + b + c + d \\ &= 9(111a + 11b + c) + (a + b + c + d) \end{aligned}$$

Now, $9(111a + 11b + c)$ is divisible by 9, since 9 is a factor. Therefore, if $(a + b + c + d)$ is also divisible by 9, the entire number is divisible by 9, as can be shown by the distributive property. Hence our rule that any number is divisible by 9 if the sum of its digits is divisible by 9.

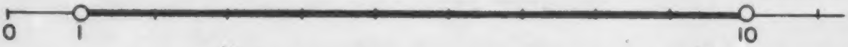
5. (a) $2x$ belongs to the set with graph



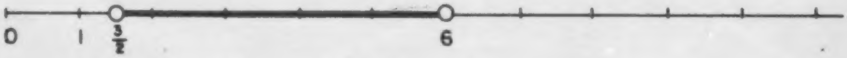
$x + 1$ belongs to the set with graph



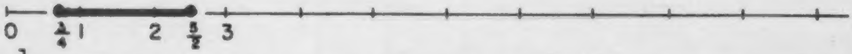
(b) 2x belongs to the set of numbers between 1 and 10.



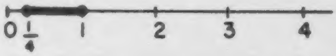
x + 1 belongs to the set of numbers between $\frac{3}{2}$ and 6.



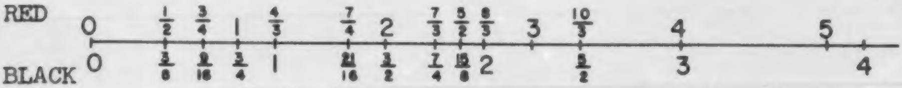
(c) x belongs to the set with graph


0
$\frac{3}{4}$
1
2
$\frac{5}{2}$
3

(d) $\frac{1}{x}$ belongs to the set with graph


0
$\frac{1}{4}$
1
2
3
4

6.


RED
0
$\frac{1}{2}$
$\frac{3}{4}$
1
$\frac{4}{3}$
$\frac{7}{4}$
2
$\frac{7}{3}$
$\frac{5}{2}$
$\frac{8}{3}$
3
$\frac{10}{3}$
4
5
BLACK
0
$\frac{3}{8}$
$\frac{9}{16}$
$\frac{3}{4}$
1
$\frac{21}{16}$
$\frac{3}{2}$
$\frac{7}{4}$
$\frac{15}{8}$
2
$\frac{5}{2}$
3
4

(a)

BLACK	RED
1	$\frac{4}{3}$
2	$\frac{8}{3}$
$\frac{5}{2}$	$\frac{10}{3}$
$\frac{7}{4}$	$\frac{7}{3}$
10	$\frac{40}{3}$
20	$\frac{80}{3}$

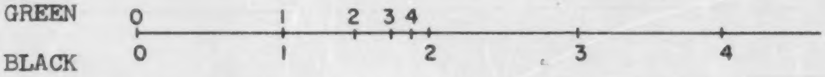
(b)

RED	BLACK
1	$\frac{3}{4}$
2	$\frac{3}{2}$
$\frac{5}{2}$	$\frac{15}{8}$
$\frac{7}{4}$	$\frac{21}{16}$
$\frac{3}{4}$	$\frac{9}{16}$
$\frac{1}{2}$	$\frac{3}{8}$

(c) No; none to the right of 0, since each red coordinate is four-thirds the corresponding black coordinate.

(d) $r = \frac{4}{3}b$, or $b = \frac{3}{4}r$.

7.


GREEN
0
1
2
3
4
BLACK
0
1
2
3
4

(a) The black coordinate of the point with green coordinate 3 is $1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$. Yes; every whole number is the green coordinate of a point.

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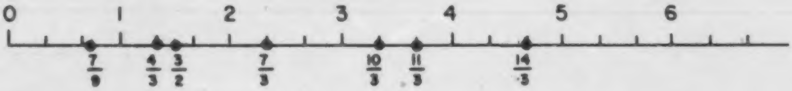
(b) There is no green coordinate for the point with black coordinate 3. There are no green coordinates for points to the right of black 2. Moreover, black 2 has no corresponding green coordinate.

(c) The point would have black coordinate

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{127}{64}.$$

8. (a)



From 1 to 2 : $\frac{4}{3}$

From 2 to 3 : $\frac{7}{3}$

From 3 to 5 : $\frac{11}{3}$

From 4 to 6 : $\frac{14}{3}$

From 5 to 8 : 6

From 1 to 8 : $\frac{10}{3}$

From 1 to $\frac{5}{2}$: $\frac{3}{2}$

From $\frac{1}{2}$ to $\frac{4}{3}$: $\frac{7}{9}$

(b) $c = a + \frac{b - a}{3}$, or $c = \frac{2a + b}{3}$.

(c) $d = a + \frac{2}{3}(b - a)$, or $d = \frac{a + 2b}{3}$.

9. This problem reviews sums of pairs of elements of a set; it is not a problem set up primarily to get an answer. The pupil who tries to write an open sentence will find he is wasting his time. Instead he should observe that the man has a set of four elements: {1.69, 1.95, 2.65, 3.15} and that he should examine the set of all possible sums of pairs of elements of the set.

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+	1.69	1.95	2.65	3.15
1.69	3.38	3.64	4.34	4.84
1.95	3.64	3.90	4.60	5.10
2.65	4.34	4.60	5.30	5.80
3.15	4.84	5.10	5.80	6.30

This is the set: {3.38, 3.64, 3.90, 4.34, 4.60, 4.84, 5.10, 5.30, 5.80, 6.30}

From this we see:

(a) The smallest amount of change he could have is 5.00 - 4.84, or 16 cents.

(b) The greatest amount of change possible is 5.00 - 3.38, or $1.62.

(c) There are four pairs of two boxes he cannot afford: one of $1.95 and one of $3.15; one of $2.65 and one of $3.15; two of $2.65; two of $3.15.

10. We can write a numeral in powers other than 10, and 8 is as good as any other. For the "8 scale" we need the set of digits {0, 1, 2, 3,···, 7}. "8" would be written "10". (Read this "one - oh.")

$$\begin{aligned} 2357_{eight} &= 2(8)^3 + 3(8)^2 + 5(8) + 7 \\ &= 1024 + 192 + 40 + 7 \\ &= 1263_{ten} \end{aligned}$$

$$\begin{aligned} 207_{ten} &= 3(64) + 1(8) + 7 \\ &= 3(8)^2 + 1(8) + 7 \\ &= 317_{eight} \end{aligned}$$

In case the pupils wish more practice in changing bases we suggest the following:

(a) $137_{ten} = 211_{eight}$

(b) $3452_{eight} = 1834_{ten}$

(c) $2345_{ten} = 4451_{eight}$

(d) $67562_{eight} = 28{,}530_{ten}$

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11. The set of all girls with 2 heads is the null set. We cannot add the number 1 to the null set since it is not 0.

12. This problem was inserted to provide pleasurable experience in reading directions and in translating. Each of the numerals from 0 to 9 represents several letters of the alphabet. The pupil translates first from numbers to letters, then from letters to numbers. Each translation, and in particular the second kind, involves a choice based on reasoning. Probably the pupils will all accept "Jane is home" for "$9034 = 7424$." This is a correct translation. However, in case some pupil protests that "$9034 = 7424$" is not a true sentence, and hence Jane is not home, accept the suggestion as a possibility. Nevertheless, make it clear that the translation of "$=$" is "is" and not "is not." Some exceptionally eager students can be encouraged to devise their own codes or problems using letters for numbers and vice versa.

(a) A possible translation is "he is hungry."

(c)
$$\begin{aligned} 3(1(10) + 2) &= 4(10) + 6 \\ 3(10 + 2) &= 40 + 6 \\ 36 &= 46 \end{aligned}$$
This sentence is not true.

(d) $7 + 2 = 2 + 7$. This sentence is true. This points toward the commutative property of addition, whose truth, as it applies here to numbers, the pupil will probably accept readily.

(e) $p(h)$ indicates the multiplication of h by p, hence $5(7) = 7(5)$. This is a true sentence. (This points toward the commutative property of multiplication.) Therefore, the sentence $p(h) > r(f)$ is not a true sentence.

(f) $6 + 5 \neq 5(6)$ is true.

(g) $8(m)$ indicates that m is to be multiplied by 8. Hence, we have $8(2) = 2(8)$. This is a true sentence. (Here again we use the commutative property of multiplication.)

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13. I does not refer back to the code used for the previous problems. Here we are to select numbers for the letters which will make the addition correct, being careful that no number is represented by two different letters. There are many solutions, such as

$$\begin{array}{r} \text{I} \\ \text{AM} \\ \text{THE} \\ \underline{\text{BOSS}} \end{array} \qquad \begin{array}{r} 5 \\ 46 \\ \underline{982} \\ 1033 \end{array}$$

$$\begin{array}{r} \text{SEND} \\ \underline{\text{MORE}} \\ \text{MONEY} \end{array}$$

has a unique solution. M must be 1, hence O must be zero. S must be either 8 or 9, but inspection shows that 8 is impossible, so S is 9. Consideration of the second and third columns shows that R is 8. Then N must be E + 1. Since 0, 1, 8 and 9 have been used,

$$N \neq 3 \text{ for } E \neq 2$$
$$N \neq 4 \text{ for } E \neq 3$$
$$N \neq 5 \text{ for } E \neq 4$$

so we let N be 6 and E be 5. Now we have used 0, 1, 5, 6, 8, 9. D and Y may be chosen from 2, 3, 4, 7. Since the sum of D and 5 must be more than 10, $D \neq 2$, $D \neq 3$, and $D \neq 4$. Therefore D is 7 and Y is 2.

This problem involves quite a bit of reasoning for ninth graders but there are always some of them who will work at it until the problem is solved. Please do not spoil their pleasure, but let them reason out the solution under their own power. We hope to give them many opportunities at this level.

14. 8432 + 1567 = 9999
2765 + 7234 = 9999
3961 + 6038 = 9999
3(9999) = 30,000 - 3
30,000 + 4028 = 34028
30,000 - 3 + 4028 = 34025.
The teacher was correct.

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Another way of writing the second problem might be:

$$\begin{array}{r} 8025\\ 4567\\ 3902\\ 5678\\ 1974\\ 5432\\ \underline{6097}\\ 35675 \end{array}$$

Here again we have 3 sums which total 30,000 - 3. It is easy to add 5,678 to 30,000 and subtract 3.

There is a nice extension of this to 5 and 6 numbers which the pupil might try. The game may also be played with numbers having more or less than 4 digits. Pupils might like to try this also.

15.

x	12	10	2	$\frac{5}{4}$	$\frac{3}{4}$
12	144	120	24	15	9
10	120	100	20	$\frac{25}{2}$	$\frac{15}{2}$
2	24	20	4	$\frac{5}{2}$	$\frac{3}{2}$
$\frac{5}{4}$	15	$\frac{25}{2}$	$\frac{5}{2}$	$\frac{25}{16}$	$\frac{15}{16}$
$\frac{3}{4}$	9	$\frac{15}{2}$	$\frac{3}{2}$	$\frac{15}{16}$	$\frac{9}{16}$

With the facts already given, the table can be completed by performing a single multiplication: $10(10) = 100$. The other spaces can be filled as follows: Use the commutative property to complete the last two rows and columns. Fill in the row and the column for 2 by use of the distributive property. (For example: $12(2) = 12(\frac{5}{4} + \frac{3}{4}) = 15 + 9 = 24$. Now comes $10(10) = 100$. Then $12(10) = 10(10) + 2(10) = 100 + 20 = 120$. Finally, $12(12) = (10 + 2)(10 + 2) = 10(10 + 2) + 2(10 + 2) = 100 + 20 + 20 + 4 = 144$.

16. The set S includes at least the set of whole numbers greater than or equal to 2. Since 2 is not specified as the smallest element of the set we cannot be certain of its lower bound.

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17. (a) $a \circ b = 2a + b$ and
$b \circ a = 2b + a$.
If $a = 0$ and $b = 1$, then $2a + b = 1$ while $2b + a = 2$ so we can see that $2a + b$ and $2b + a$ do not name the same number for all a and b. The operation is not commutative.

(b) $a \circ b = \frac{a + b}{2}$ and $b \circ a = \frac{b + a}{2}$.
Since $a + b = b + a$, we can see that
$\frac{a + b}{2}$ and $\frac{b + a}{2}$ name the same number for all a and b. The operation is commutative.

(c) $a \circ b = (a - a)b$ and
$b \circ a = (b - b)a$
Since $(a - a)b$ and $(b - b)a$ name the same number (0) for all a and b, the operation is commutative.

(d) $a \circ b = a + \frac{1}{3}b$ and
$b \circ a = b + \frac{1}{3}a$.
If $a = 6$ and $b = 3$, then $a + \frac{1}{3}b = 7$ while $b + \frac{1}{3}a = 5$; so we can see that $a + \frac{1}{3}b$ and $b + \frac{1}{3}a$ do not name the same number for every a and every b. The operation is not commutative.

(e) $a \circ b = (a + 1)(b + 1)$ and
$b \circ a = (b + 1)(a + 1)$
Since $(a + 1)(b + 1)$ and $(b + 1)(a + 1)$ name the same number for all a and b, the operation is commutative.

18. (a) $(a \circ b) \circ c = 2(2a + b) + c = 4a + 2b + c$.
$a \circ (b \circ c) = 2a + 2b + c$.
If $a = 1$, $b = 1$, $c = 2$, then $4a + 2b + c = 8$ while $2a + 2b + c = 6$; so we can see that $4a + 2b + c$ and $2a + 2b + c$ do not name the same number for all a, b, and c. The operation is not associative.

(b) $(a \circ b) \circ c = \frac{a + b + 2c}{4}$ and

$a \circ (b \circ c) = \frac{2a + b + c}{4}$

So if $a = 1, b = 0, c = 2$, then $\frac{a + b + 2c}{4} = \frac{5}{4}$ while $\frac{2a + b + c}{4} = 1$ and we can see that $\frac{a + b + 2c}{4}$ and $\frac{2a + b + c}{4}$ do not name the same number for all a, b, and c. The operation is not associative.

(c) $(a \circ b) \circ c = 0$ and

$a \circ (b \circ c) = 0$

Since the same number (0) is named for all a, b, and c, the operation is associative.

(d) $(a \circ b) \circ c = a + \frac{1}{3}b + \frac{1}{3}c$ and

$a \circ (b \circ c) = a + \frac{1}{3}b + \frac{1}{9}c$

It is clear these always differ by $\frac{2}{9}c$ so $a + \frac{1}{3}b + \frac{1}{3}c$ and $a + \frac{1}{3}b + \frac{1}{9}c$ do not name the same number for all a, b, and c. The operation is not associative.

(e) $(a \circ b) \circ c = (ab + b + a + 2)(c + 1)$ and $a \circ (b \circ c) = (a + 1)(bc + c + b + 2)$. So if $a = 0$, $b = 1$, $c = 1$, the first expression is 6 while the second is 5. Thus we can see that the expressions do not name the same number for all a,b and c. The operation is not associative.

19. Let x be the number of days it would take the two men together to paint the house. The first man can paint $\frac{1}{5}$ of the house in one day. The second man can paint $\frac{1}{3}$ of the house in one day. Together the two men can paint $\frac{1}{x}$ of the house in one day.

The open sentence is $\frac{1}{3} + \frac{1}{5} = \frac{1}{x}$

$$\frac{8}{15} = \frac{1}{x}$$

$$x = \frac{15}{8}$$

The truth set is $\{\frac{15}{8}\}$.

The two men save $3\frac{1}{8}$ days by working together instead of the first man working alone.

$\frac{\frac{25}{8}}{\frac{15}{8}}$ or $\frac{5}{3}$ days is the time saved each day the two men work together instead of the slower man working alone. For example, if a job would take the two men working together 3 days, the first man could do it in 8 days. The saving in time is $\frac{5}{3}$ of 3 or 5 days.

20. Let x be the number of hours it would take the combination of pipes to fill the tank. One pipe can fill $\frac{1}{5}$ of the tank in one hour. The second can fill $\frac{1}{3}$ of the tank in one hour. And the third can drain $\frac{1}{4}$ of the tank in one hour. Working together the pipes can fill $\frac{1}{x}$ of the tank in one hour. The open sentence is

$$\begin{aligned} \frac{1}{5} + \frac{1}{3} - \frac{1}{4} &= \frac{1}{x} \\ (12 + 20 - 15)x &= 60 \\ x &= \frac{60}{17} \end{aligned}$$

The truth set is $\{\frac{60}{17}\}$.

The tank will be filled in $\frac{60}{17}$ hours if all pipes are left open. After $\frac{60}{17}$ hours the tank will start to overflow.

Chapter 6

THE REAL NUMBERS

In Chapters 1 to 5 the student has been discovering and applying properties of operations on a set of numbers. This set consists of zero and the numbers assigned to the points to the right of zero on the number line. His work with familiar numbers gave him security with such concepts as the associative, commutative, and distributive properties, open sentences, truth sets, etc.

With this background, he is now ready to give names to numbers which we assign to points to the left of zero on the number line. The total set of numbers corresponding to all points of the line, the set of real numbers, is now his field of activity.

In Chapter 6 we attempt to familiarize the student with the total set of real numbers. This includes the order of real numbers, comparison of real numbers, and the operation of determining the opposite of a real number. The final section is devoted to a definition and discussion of the absolute value of a real number.

In general a system of numbers is a set of numbers and the operations on these numbers. Hence, we do not have the real number system until we define the operations of addition and multiplication for real numbers. This is done in Chapter 7 (addition) and Chapter 8 (multiplication). Our point of view is that the operations must be extended from the non-negative real numbers to all real numbers. Thus the definitions of addition and multiplication must be formulated exclusively in terms of non-negative numbers and operations (including taking opposites) on them. It is essential, of course, that the fundamental properties of these operations be preserved in this so-called extension process.

Order in the real numbers is introduced in Chapter 6. In Chapter 9 we return to order, but with an important shift in our point of view. Previously we have tended to use order as a convenient way to discuss certain aspects of numbers. In this sense "<" and ">" were simply fragments of language. In Chapter 9 we treat "<" as an order relation having specific mathematical properties in its own right.

Chapter 10 deals with subtraction and division. These operations are defined in terms of addition and multiplication. In this sense we retain the notion that the real number system is a structure which may be developed in terms of two basic operations.

It should be mentioned that in this course we have chosen to approach the negative numbers in a manner different from some writers. Instead of presenting a new set of numbers (the real numbers) and then identifying a particular subset of these (the non-negative) with the original set (the numbers of arithmetic), we have chosen the following approach. We extend the numbers of arithmetic to the set of real numbers by attaching the negative numbers to the familiar numbers of arithmetic. This has several advantages: First, we do not need to distinguish between "signed" and "unsigned" numbers; to us the non-negative real numbers are the numbers of arithmetic. Second, it is not necessary for us to prove that the familiar properties hold for the non-negatives, for these properties are carried over intact along with the numbers of arithmetic. In this manner, we avoid the confusion of establishing an "isomorphism" between positive numbers and "unsigned numbers". Notice that we have no need whatsoever for the ambiguous word "sign".

In general, we have taken the point of view that a ninth grade student really has some experience with negative numbers. He is quite ready to label the points to the left of 0 and, in so doing, make the extension to which we referred.

The treatment of absolute value in this chapter exemplifies what has been referred to as the "spiral technique". The introduction to absolute value is followed in each succeeding chapter by more and more uses at different levels of abstraction. Thus the teacher need not give a full development of this topic in Chapter 6 since it will reappear regularly in later portions of the book.

6-1. The Real Numbers.

We introduce the negative numbers in much the same way that we labeled the points on the right side of the number line, which correspond to the positive real numbers. Our notation for negative four, for example, is $^{-}4$, and we definitely intend that the dash "-" be written in a raised position. At this stage, we do not want the student to think that something has been done to the number 4 to get the number $^{-}4$, but rather that $^{-}4$ is a name of the number which is assigned to the point 4 units to the left of 0 on the number line.

In Section 6-3, the student will be able to think of $^{-}4$ as the number obtained from 4 by an operation called "oppositing". The opposite of 4 will be symbolized as -4, the dash being written in a lowered position, and -4 will turn out to be a more convenient name for $^{-}4$.

Since each number of arithmetic has many names, so does each negative real number. For example, the number $^{-}7$ has the names $^{-}(\frac{14}{2})$, $^{-}(7 \times 1)$, etc.

In drawing the graph of real numbers, the student should be aware that the number line picture is only an approximation to the true number line. Consequently, any information which he deduces from his number line picture is only as accurate as his drawing.

Once the negative numbers have been introduced, we introduce integers and the entire set of rational numbers. We introduce irrational numbers only so that we can talk about real numbers and the real number line.

We could call this set the "set of numbers", but some students may learn about complex numbers later on. We do not want the teacher to discuss these complex numbers now but the students should be aware that there are numbers other than those we have called real.

A common misunderstanding is that some numbers on the line are real and others are irrational. The student should be encouraged to say, at least for the time being, that "-2" is a real number which is a rational number and a negative integer; $\frac{3}{2}$ is a real number which is a rational number; -2 is a real number which is a negative irrational number".

We want the student to be very much aware that there are infinitely many points on the number line which are not rational numbers. He will eventually learn how to name many of these, but he should not be concerned about this at present. In order that he does not jump to the conclusion that all these new numbers are simply variants of $\sqrt{2}, \sqrt{3}$, etc., we introduce an intuitive method for determining π on the number line in Exercise 2 of Problem Set 6-1c.

The idea of rolling a circle along the number line to determine π can, of course, be used to "locate" numbers like $\sqrt{2}\,\pi$ by considering the circle to have diameter $\sqrt{2}$.

The number π is quite different in character from numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{\sqrt{5}}$, $\sqrt{7 - 4\sqrt{3}}$, etc. All these latter numbers are solutions to equations of the form

$$a_0 + a_1x + a_2x^2 + \ldots + a_mx^m = 0$$

in which $a_0, a_1, a_2, \ldots, a_m$ are integers. For example, the number $\sqrt{3}$ satisfies the equation $3 - x^2 = 0$, and $\sqrt{7 - 4\sqrt{3}}$ is a solution of

$$1 - 14x^2 + x^4 = 0.$$

However, π satisfies no such equation. It is an example of what is called a transcendental number, with numbers like $\sqrt{2}$, $\sqrt{\sqrt{5}}$, etc., being called algebraic numbers.

It might be pointed out to the student who is inquisitive about irrational numbers that these numbers differ in an interesting way from rational numbers in their decimal representation. Any rational number can be represented by a repeating decimal. Some examples are:

$\frac{1}{4} = .25000\ldots$ (usually written .25)

$\frac{1}{7} = .142857142857\ldots$

$\frac{29}{99} = .292929\ldots$

The decimal representation of any irrational number, such as π, $\sqrt{2}$, $\sqrt[3]{4}$, etc., is an infinite non-repeating decimal.

Answers to Oral Exercises 6-1a; page 198:

Answers may vary for questions 1-4. Any five elements of the listed set are satisfactory.

1. $\{1, 2, 3, 4, 5, \ldots\}$
2. $\{^-1, ^-2, ^-3, ^-4, ^-5, \ldots\}$

3. {0, 1, 2, 3, 4, ... }

4. {1, 2, 3, 4, 5, ... }

5. The empty set is the set which has no elements.

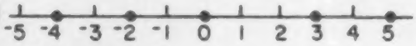
Answers to Problem Set 6-1a; pages 198-199:

1. (a) W = {0, 1, 2, 3, ... }
   P = {1, 2, 3, 4, ... }
   L = {0, 1, 2, 3, ... }
   I = {..., $^{-}2$, $^{-}1$, 0, 1, 2, ... }
   N = {1, 2, 3, 4, ... }
   Q = {0, $^{-}1$, $^{-}2$, $^{-}3$, ... }
   S = {$^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$, ... }

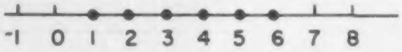
   (b) W and L are the same.
   P and N are the same.

   (c) All are subsets of I.
   Q and S are subsets of Q.
   L, W, P and N are subsets of L.
   P and N are subsets of P.

2. (a)



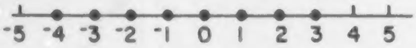
(b)



(c)

the empty set

(d)



(e)

the empty set

3. (a) 3 is to the right of $^{-}4$
   (b) 5 " $^{-}4$
   (c) $^{-}2$ " $^{-}4$
   (d) 0 " $^{-}2$
   (e) 0 " $^{-}4$
   (f) 5 " 0

4. (a) If x is the number of pigeons Bill had 3 years ago, then the number he has now is 2x + 25.

$$2x + 25 = 77$$

(b) If x is the rate in miles per hour of the first train, then the rate in miles per hour of the second train is $2x + 10$.

$$4x + 4(2x + 10) = 340$$

(c) Let w be the width in inches.

$$2w + 2(62) = 196$$

or

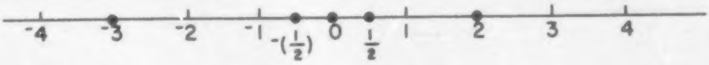
$$2w + 124 = 196$$

__Answers to Oral Exercises 6-1b__; pages 200-201:

1. $I = \{^-5, ^-4, ^-1, 0, 1, 2, 6\}$
2. $W = \{0, 1, 2, 6\}$
3. $A = \{1, 2, 6\}$
4. $P = \{^-5, ^-4, ^-1\}$
5. $N = \{1, 2, 6\}$
6. $G = \{\frac{1}{4}, 1, \frac{3}{2}, 2, \frac{7}{3}, 6\}$
7. $L = \{^-5, ^-4, ^-(\frac{10}{3}), ^-(\frac{3}{2}), ^-1\}$
8. $Y = \{0, 1, 2, 6\}$

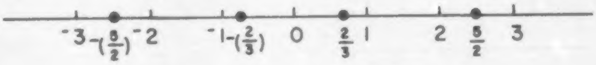
__Answers to Problem Set 6-1b__; page 201:

1. (a)



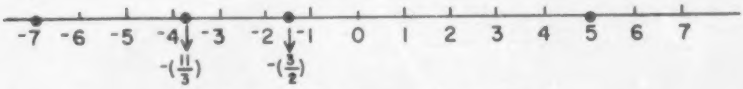
⁻4
⁻3
⁻2
⁻1
⁻(½)
0
½
1
2
3
4

(b)



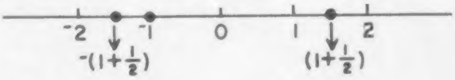
⁻3
⁻(5/2)
⁻2
⁻1
⁻(2/3)
0
2/3
1
2
5/2
3

(c)



⁻7
⁻6
⁻5
⁻4
⁻3
⁻2
⁻1
0
1
2
3
4
5
6
7
⁻(11/3)
⁻(3/2)

(d)



2. (a) 5 is to the right of 4.
   (b) 0 " $^-5$.
   (c) $^-4$ " $^-7$.
   (d) 1 " $^-(\frac{1}{2})$.
   (e) Same point
   (f) $^-(\frac{15}{4})$ is to the right of $^-4$.
   (g) $^-(\frac{21}{4})$ " $^-(\frac{16}{3})$.

Here we are building toward the ordering of numbers again, which we will develop further in the next section of this chapter.

3. If w is the number of inches in the width, then $4w$ is the number of inches in the length. The perimeter is 24 inches.

$$w + 4w + w + 4w = 24$$

or

$$10w = 24$$

(truth set $\{2.4\}$
answer: $9\frac{3}{5}$ inches)

Answers to Problem Set 6-1c; pages 203-204:

1. (a) $^-2$ is an integer, rational, real.

   (b) $^-(\frac{10}{3})$ is rational, real.

   (c) $\sqrt{2}$ is a real number.

   (d) 0 is whole, a real number, an integer, a rational number.

2. (a) False
   (b) True
   (c) True
   (d) False
   (e) True
   (f) True
   (g) True
   (h) False

3. π, $^-\pi$. π is between 3 and 4. $^-\pi$ is between $^-3$ and $^-4$.

4. (a) Three
   (b) Seven

5. (a) If s is the number of years in sister's age, then $2s$ is the number in brother's age and Mary is $2(2s)$ years old.

$$2(2s) + 2s + s = 15$$

or

$$7s = 15$$

(truth set is $\{\frac{15}{7}\}$
answer is: sister $2\frac{1}{7}$ years old
brother $4\frac{2}{7}$ years old
Mary $8\frac{4}{7}$ years old)

   (b) If q represents the rate of travel of one boy, $2q + 2(2q) = 90$.

   (c) If x represents the first integer, $x + (x + 2) = 86$.

6-2. Order on the Real Number Line.

We believe that the student will expect the relation "is greater than" for the real numbers to have the same meaning as it did for what we now call the non-negative real numbers.

Although "is greater than" was defined to mean "is to the right of" on the number line for the positive numbers, it could clearly be interpreted as "is farther from zero than". It is then plausible that">" for the real numbers might well have this latter meaning. On the other hand, the example of the thermometer does not agree with this interpretation, nor would such familiar things as the variation in the height of tides or elevations above and below sea level.

There is also a good mathematical reason for rejecting this plausible interpretation. The mathematician is never really interested in a relation as such, but rather in the properties it enjoys. Whatever meaning is attached to "is greater than" we want to be able to say, for example, that precisely one of the sentences "$3 > {}^{-}3$" and "${}^{-}3 > 3$" is true. This plausible interpretation does not permit this comparison, since neither ${}^{-}3$ nor 3 is farther from zero than the other. Here we choose to retain the interpretation ">" to mean "is to the right of" on the number line.

The comparison property here given is also called the trichotomy property of <. Notice that it is a property of <; that is, given any two numbers, they can be ordered so that one is less than the other. When the property is stated using numerals, we must include the third possibility that the numerals name the same number. Hence, the name "trichotomy".

Although "$a < b$" and "$b > a$" involve different orders, these sentences say exactly the same thing about the numbers a and b. Thus, we can state a trichotomy property of > as:

> For any number a and any number b,
> exactly one of these is true:
> $a > b$, $a = b$, $b > a$.

If instead of concentrating attention on the order relation, we concentrate on the two numbers, then either "$a > b$" or "$a < b$" is true, but not both. Here we fix the numbers a and b and then make a decision as to which order relation applies. It is purely a matter of which we are interested in: the numbers or the order. The comparison property is concerned with an order.

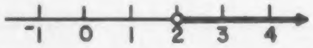
Answers to Oral Exercises 6-2a; page 206:

1. By "is less than" we shall mean "is to the left of" on the number line.
2. This, "$\geq$", means "is to the right of or equal to" on the number line.
3. This, "$\leq$", means "is to the left of or equal to" on the number line.

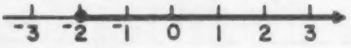
Answers to Problem Set 6-2a; pages 206-207:

1. (a) False
   (b) True
   (c) True
   (d) False
   (e) True
   (f) False
   (g) True
   (h) False
   (i) True
   (j) True

2. (a)



(b)



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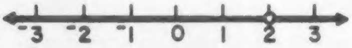
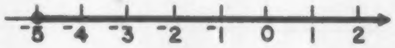
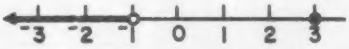


(d)



(e)



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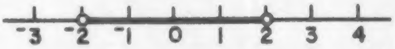


3. (a) $\frac{3}{5} = \frac{6}{10}$
   (b) $\frac{3}{5} > \frac{3}{6}$
   (c) $\frac{9}{12} > \frac{8}{12}$
   (d) $\frac{9}{12} > \frac{4}{6}$
   (e) ${}^{-}(\frac{3}{5}) < \frac{3}{6}$
   (f) ${}^{-}(\frac{3}{5}) > {}^{-}(\frac{4}{5})$
   (g) $(\frac{{}^{-}3}{5}) < {}^{-}(\frac{3}{6})$

4. (a) 6 degrees
   (b) to ${}^{-}5$ degrees
   (c) 50 degrees

Do these on the number line as preparation for the addition of real numbers that is developed in the next chapter.

5. (a) $n \geq {}^-18$
   (b) $a > b$
   (c) $n = x + 5$
   (d) $n - 7 < n + 4$
   (e) $n + 5 < n + 8$
   (f) $n - 5 = 2n - 4$
   (g) $3n = 8$ or $n - 4 > 2$
   (h) $n = {}^-4 - 6$

Answers to Problem Set 6-2b; pages 208-209:

1. (a) $>$
   (b) $>$
   (c) $>$
   (d) $=$
   (e) $<$
   (f) $>$
   (g) $<$
   (h) $>$
   (i) $>$
   (j) $<$

2. (a) $5 < 6$
   (b) ${}^-3 < 0$
   (c) ${}^-(\frac{2}{3}) < {}^-(\frac{5}{9})$
   (d) $(1.4)^2 < 2$
   (e) $3 < \pi$

3. (a) ${}^-5 > {}^-7$
   (b) $0 > {}^-8$
   (c) $8 > 0$
   (d) $\frac{1}{3} > .3$
   (e) $\frac{19}{8} > 2$

4. (a) If the original price was p dollars, then the discount was $\frac{1}{3}p$ and the sale price was $p - \frac{1}{3}p$.
   So $p - \frac{1}{3}p = 33$ or $\frac{2}{3}p = 33$. (Answer: the original price was \$49.50)
   (b) If x represents the brother's age,
   $2x - 4 = 12$. $(x = 8)$
   (c) If d represents the height of the box,
   $8 \cdot 12 \cdot d = 864$. $(d = 9)$

6-3. Opposites.

Your students have observed by now that, except for zero, the real numbers occur in pairs, the two numbers of each pair being equidistant from zero on the real number line. Each number in such a pair is called the opposite of the other. To complete the picture, zero is defined to be its own opposite.

On locating the opposite of a given number on the number line, you may want to use a compass to emphasize that the number and its opposite are equidistant from zero.

It is clearly much too tedious to have to write "the opposite of 2", "the opposite of $^{-}(\frac{1}{2})$", etc. By having the students write down a few such phrases, we hope to suggest to them that a shorthand is needed. The lower dash " - " which we use is perhaps the most suggestive device to indicate the opposite of a given number and we are very quick to observe that, for example $^{-}2$ and -2 are two different names for the same number.

Having observed that each negative number is also the opposite of a positive number, it is apparent that we have no need for two symbolisms to denote the negative numbers. Since the lower dash " - " is applicable to numerals for all real numbers while the upper dash "$^{-}$" has significance only when attached to numerals for positive numbers, we naturally retain the lower dash. There are other less important reasons for dropping the upper dash in favor of the lower: it is easier to write, say, -5 than $^{-}5$; more care must be used in denoting negative fractions with the upper dash than with the lower (for example, $\frac{^{-}12}{5}$ could be misread as $\frac{-12}{5}$); the lower dash is universally used, etc. Henceforth, then, negative numbers like $^{-}5$, $^{-}(\frac{3}{7})$, $^{-}2$, etc. will be written as -5, $-\frac{3}{7}$, -2, etc.

The student must learn to designate the opposite of a given number by means of the definition. The student should not be permitted to say, "To find the opposite of a number, change its sign". This is very imprecise (in fact, we have never attached a "sign" to the positive numbers) and will lead to a purely manipulative algebra which we want to avoid at all costs.

The student is well aware that the lower dash " - " is read "minus" in the case of subtraction. We prefer to retain the word "minus" for the operation of subtraction and not use it as an alternate word for "opposite of". Thus the dash attached to a variable, such as "$-x$" will be read "opposite of".

The opposite of the opposite of the opposite of a number is the opposite of that number. What is the opposite of the opposite of a negative number? The (negative) number, of course!

If x is a positive number, then $-x$ is a negative number. The opposite of any negative number x is a positive number $-x$. And $-0 = 0$. Thus, the student should not jump to the conclusion that when n is a real number, then $-n$ is a negative number; this is true only when n is a positive number. Note the emphasis here on the use of " - " as "the opposite". Because of complications that will arise in later work regarding "-a" which the students insist on calling a "negative number a", it is worthwhile to reemphasize the meaning of the upper dash (read "negative") as meaning "to the left of zero" or "less than 0", while the middle dash (read "opposite of") means "on the opposite side of zero".

There are no oral exercises for 6-3a since the problem set given might just as well be done all or in part orally.

Answers to Problem Set 6-3a; page 211:

1. (a) $^{-}55$, negative 55
   (b) 55
   (c) $^{-}33.5$, negative 33.5
   (d) 0
   (e) 100
   (f) $\frac{3}{4}$
   (g) $^{-}(\frac{2}{3})$, negative $\frac{2}{3}$
   (h) $2\frac{1}{2}$
   (i) $^{-}2\frac{1}{2}$, negative $2\frac{1}{2}$
   (j) $^{-}1,000,000,000$, negative 1 billion
   (k) 1,000,000,000
   (l) $^{-}8$, negative 8
   (m) $^{-}9$, negative 9
   (n) $^{-}16$, negative 16

2. The only true statement is "The opposite of a positive number is a negative number".

3. The only true statement is "The opposite of a negative number is a positive number".

4. The opposite of zero is zero.

5. "Negative 9" and "the opposite of 9" are names for the same number. The first says "9 units to the left of 0", the second says "the number which corresponds to the point which is the same distance from 0 as 9 is, but on the opposite side of 0".

Answers to Oral Exercises 6-3b; page 213:

1. (a) 20
   (b) $^{-}20$
   (c) $^{-}(\frac{1}{2})$
   (d) 5
   (e) 210
   (f) $^{-}37.5$

Answers to Problem Set 6-3b; page 213:

1. (a) 40
   (b) $^{-}(\frac{7}{2})$
   (c) $^{-}8$
   (d) 8
   (e) 0
   (f) $^{-}9$

2. If y is a positive number, then -y is a negative number.

3. If y is a negative number, then -y is a positive number.

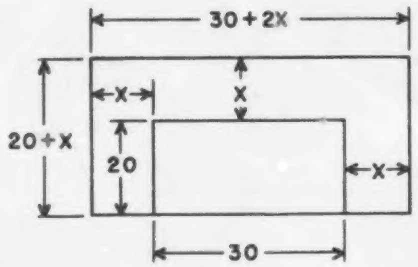
4. If y is 0, then -y is 0.

5. If -y is positive, then y is negative.

6. If -y is negative, then y is positive.

7. If -y is 0, then y is 0.

8.



If x is the number of feet in the width of the walk, then the length is 30 + 2x, the width is 20 + x. The perimeter is the sum of the sides.

$(20 + x) + (20 + x) + (30 + 2x) + (30 + 2x) = 150$

or $100 + 6x = 150$

(truth set $\{\frac{25}{3}\}$ Answer: $8\frac{1}{3}$ feet is width of walk.)

Answers to Oral Exercises 6-3c; page 216:

1. (a) $2.97 > -2.97$ $\quad 2.97 > -2.97$
   (b) $2 > -12$ $\quad 12 > -2$
   (c) $-358 > -762$ $\quad 762 > 358$
   (d) $1 > -1$ $\quad 1 > -1$
   (e) $-121 > -370$ $\quad 370 > 121$
   (f) $.24 > .12$ $\quad -.12 > -.24$
   (g) $0 = -0$ Zero is the only number with the property that it is equal to its opposite.
   (h) $-.01 > -.1$ $\quad .1 > .01$

(i) $.1 > .01$ $-.01 > -.1$

2. This means that $x > 3$ or $x < 3$.

Answers to Problem Set 6-3c; pages 217-218:

1. (a) $-1 < 3$, $-3 < 1$

   (b) $^{-}(\frac{1}{2}) < \frac{3}{4}$, $-(\frac{3}{4}) < -(^{-}\frac{1}{2})$

   (c) $^{-}(\frac{1}{6}) < \frac{2}{7}$, $-(\frac{2}{7}) < -(^{-}\frac{1}{6})$

   (d) $^{-}\pi < \sqrt{2}$, $-\sqrt{2} < -\,^{-}\pi$

   (e) $\pi < \frac{22}{7}$, $-(\frac{22}{7}) < -\pi$

   (f) $3(\frac{4}{3} + 2) < \frac{5}{4}(20 + 8)$, $-\left(\frac{5}{4}(20 + 8)\right) < -\left(3(\frac{4}{3} + 2)\right)$

   (g) $^{-}(5 + 4) < 2(8 + 5)$, $-\left(2(8 + 5)\right) < -\,^{-}(5 + 4)$

   (h) the same number

2. (a) $-(^{-}7.2)$

   (b) 3

   (c) $-(^{-}5)$

   (d) $-(^{-}\sqrt{2})$

   (e) 17

   (f) $.01$

   (g) 2 or $-(-2)$

   (h) $\frac{9}{16}$ or $(1 - \frac{1}{4})^2$

   (i) $1 - (\frac{1}{4})^2$ or $\frac{15}{16}$

   (j) $\frac{1}{2} - \frac{1}{3}$ or $\frac{1}{6}$

Here we are building toward the meaning of "absolute value of a number" and that the greater of a number and its opposite is always the positive value.

3. (a) $x < 1$, $^{-}1 < -x$

   (b) $x > {}^{-}2$, $-(^{-}2) > -x$

   (c) $x > 0$, $0 > -x$

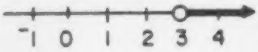
   (d) $x < 0$, $0 < -x$

   (e) $x > 1$, $^{-}1 > -x$

   (f) $x \leq {}^{-}2$, $-(^{-}2) \leq -x$

   (g) $x > {}^{-}2$ and $x < 2$,
   $2 > -x$ and $-x > -2$

4. (a)



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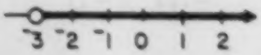

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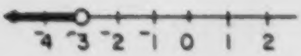

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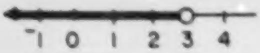


5. (a) The set of all numbers except $^-3$
   (b) The set of all numbers except 3
   (c) The set of all numbers less than 0
   (d) The set of all numbers greater than 0.
   *(e) The set of all numbers equal to or less than 0
   *(f) The set of all numbers equal to or greater than 0

6. (a) If John scored n points, then $n >$
   (b) If he has n dollars, then $n \leq 0$ an
   (c) If the original bill was d dollars, the

6-4. Absolute Value.

The concept of the absolute value of a number is one of the most useful ideas in mathematics. We will find an immediate application of absolute value when we define addition and multiplication of real numbers in Chapters 7 and 8. In Chapter 10 it is used to define distance between points; in Chapter 12 we define $\sqrt{x^2}$ as $|x|$; in Chapter 15 it will provide a good example of an equation with extraneous solutions. Through Chapters 16 to 18 absolute values are involved in open sentences in two variables and in Chapter 19 it gives us interesting examples of functions. In later mathematics courses, in particular, in the calculus and in approximation theory, the idea of absolute value is indispensable.

The usual definition of the absolute value of the real number n is that it is the number $|n|$ for which

$$|n| = \begin{cases} n, & n \geq 0 \\ -n, & n < 0. \end{cases}$$

Quite likely it is this form of the definition that is the origin of the difficulty which the students sometimes have when they first encounter absolute value. We have tried to circumvent this difficulty by defining the absolute value of a number in such a way that it can be pictured on the real number line: The absolute value of 0 is 0 and of any other real number is the greater of that number and its opposite. This implies that the absolute value of a number is 0 or a positive number.

By observing that this "greater" of a number and its opposite is just the distance between the number and 0 on the real number line, we are able to interpret the absolute value "geometrically".

Avoid at all costs allowing the student to think of absolute value as the number obtained by "dropping the sign". Such a habit leads to endless trouble when variables are involved.

It is quite apparent that the greater of a positive number and its opposite is just the number itself. Furthermore, $|0|$ is defined outright to be 0. These two statements can be expressed symbolically as:

$$\text{If } x \geq 0, \text{ then } |x| = x.$$

For negative numbers, the number line picture should convince the students that the greater of, for example, -5, $-(\frac{1}{2})$, -3.1, and -467 and their opposites 5, $\frac{1}{2}$, 3.1, and 467 are, respectively, 5, $\frac{1}{2}$, 3.1, and 467. This same picture cannot help but tell them that the greater of any negative number and its opposite is the opposite of the (negative) number. Symbolically if $x < 0$, then $|x| = -x$.

We have therefore arrived at the usual definition of absolute value.

For all real numbers x,

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Answers to Oral Exercises 6-4a; page 220:

1. (a) $|-7| = 7$
   (b) $|-(-3)| = 3$
   (c) $|(6 - 4)| = 2$
   (d) $|14 \times 0| = 0$
   (e) $|-(14 + 0)| = 14$
   (f) $|-(-(-3))| = 3$

2. If x is 3, $|x| = 3$.
   If x is -2, $|x| = 2$.

3. A non-negative number

4. A positive number

5. Yes

6. $|x|$

7. No

8. When $x = 0$

Answers to Oral Exercises 6-4b; pages 222-223:

1. $-x$

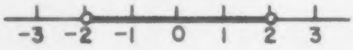
2. (a) False
   (b) True
   (c) False
   (d) True
   (e) True
   (f) True
   (g) True
   (h) True

3. (a) 5
   (b) 5
   (c) -5
   (d) -5
   (e) 5
   (f) 4
   (g) 10
   (h) -3
   (i) 1
   (j) 5
   (k) -1
   (l) -5
   (m) 2
   (n) -1
   (o) 10
   (p) -10
   (q) -10

Answers to Problem Set 6-4; page 223:

1. (a) $\{-1, 1\}$
   (b) $\{-3, 3\}$
   (c) $\{-3, 3\}$
   (d) $\{-3, 3\}$

2. (a)



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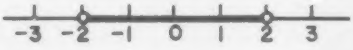

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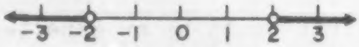
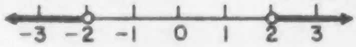


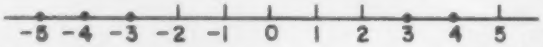
(d)



The similarities of these two pairs of graphs are worth pointing out since they provide a clue for a procedure for solving equations and inequalities with absolute value of the variable. That is, to write as two sentences with conjunction and/or as is appropriate.

3\.

etc.



-5 belongs to the set, 0 does not, -10 does, 4 does.

4. (a) $\frac{1}{2}, \frac{5}{4}, \frac{11}{5}$ (infinitely many possible answers)

(b) $-1, -\frac{3}{2}, -5$ (infinitely many possible answers)

(c) $-\frac{1}{2}, -\frac{11}{19}, -\frac{5}{12}$ (infinitely many possible answers)

(d) There are none. (e. g., the set of numbers in P but not in R is the empty set.)

5. The truth set of $|x| = 0$ is $\{0\}$.
The truth set of $|x| = -1$ is the empty set.

*6. If Pete is x years old, then Sam is $x + 3$ years old and Bob is $2x$ years old.
The father being more than twice the sum of their ages gives the sentence:

$45 > 2(x + (x + 3) + 2x)$ (This is the expected response.)

or $45 > 2(4x + 3)$

or $45 > 8x + 6$

or $8x < 39$

$x < 4\frac{7}{8}$

(Answers: Pete is less than $4\frac{7}{8}$ years old.
Sam is less than $7\frac{7}{8}$ years old.
Bob is less than $9\frac{3}{4}$ years old.)

It might be worth pointing out that we started with the smallest number in describing a variable and show the students what it would look like if we started with Bob's age.

If x = number years of Bob's age, than $\frac{x}{2}$ is the number of years of Pete's age and Sam is $(\frac{x}{2} + 3)$ years old, so the sentence becomes:

$45 > 2(\frac{x}{2} + (\frac{x}{2} + 3) + x)$. (This could be an expected response.)

$45 > x + x + 6 + 2x$

$45 > 4x + 6$

$4x < 39$

$x < 9\frac{3}{4}$, but this is not the same x as before.

Answers to Review Problem Set; pages 225-227:

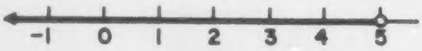
1. (a) W is a subset of I.
   (b) N is a subset of W.
   (c) N is a subset of R.
   (d) R is a subset of R*.
   (e) I is a subset of R.
   (f) P and Q are not related as subsets--they have no elements in common.
   (g) J is a subset of R*.
   (h) I and J have no elements in common.
   (i) W is a subset of R*.
   (j) Neither; P does not contain zero which is in W.

2. $a > b$ means "a is greater than b" or "a is to the right of b".

3. (a) $<$
   (b) $<$
   (c) $>$
   (d) $>$
   (e) $>$
   (f) $<$
   (g) $<$
   (h) $>$
   (i) $>$
   (j) $=$

4. π, $\sqrt{2}$, $\sqrt{5}$ etc.
   $\sqrt{16}$ is not irrational
   $\sqrt{49}$ is not irrational
   $\sqrt{5}$ is irrational

5. (a)



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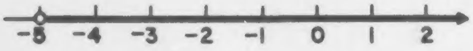

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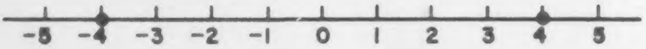

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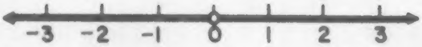
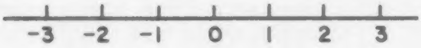
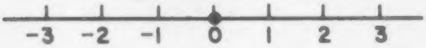
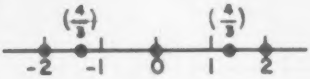
(d) 

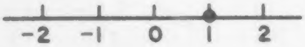
(e)  the empty set

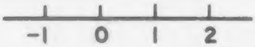
(f) 

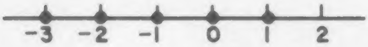
6. $a = b$ or $a < b$ or $a > b$. The comparison property of order.

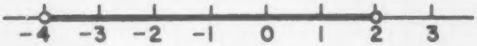
7. $-x > -3$

8. (a) 

(b) 

(c)  the empty set

(d) 

(e) 

9. (a) $x > -1$ and $x < 2$ (c) $|x| < 1$
(b) $x > -1$ (d) $|x| > 1$

10. (a) The set of all integers from -1 to 4 inclusive.
(b) The set consisting of $\frac{1}{2}$ and all numbers equal to or greater than 2 and all numbers less than -2.

11. If the number is x, then the sentence is
$x + (x + 4\frac{1}{2}) = 7.3$. (Answer: the number is 1.4)

12. If the number is x, then the sentence is
$x + 7.6x = 58$. (Answer: the number is $6\frac{32}{43}$.)

13. If the number is n, then the sentence is
$n(n + 3\frac{1}{2}) = 84$. (Answer: the number is $\frac{-7 \pm \sqrt{1393}}{4}$.)

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14. If n is the number of pages in the smaller volume, then the larger volume has $(n + 310)$ pages and the sentence is

$n + (n + 310) > 1000$. (Answer: One book has more than 345 pages and the other has more than 655 pages.)

15. If the second plane was flying at an average speed of x miles per hour, and $\frac{800}{200} = 4$ is the number of hours flown by the first plane, then the second plane flew 3 hours and the sentence is:

$3x = 800$. (Answer: $266\frac{2}{3}$ mph. is the average speed.)

Suggested Test Items

1. Determine which of the following sentences are true:

(a) $|-7| = 7$

(b) $-\frac{2}{3} \neq -|-\frac{2}{3}|$

(c) $|2| \geq |-2|$

(d) $|-5| + |-7| = 12$

(e) $-|^{-}2| = 2$

2. Rearrange the following numbers in order from the least to the greatest:

$$-\frac{1}{8},\ -2,\ -\frac{1}{3},\ 0,\ \frac{1}{2},\ -\frac{5}{2}.$$

3. In each of the following write one of the symbols $<$, $>$, or $=$ in the place indicated so that a true sentence results.

(a) 2 _____ $-|-3|$

(b) -4 _____ -7

(c) $-|-3|$ _____ -3

(d) $-\frac{14}{17}$ _____ $-\frac{15}{17}$

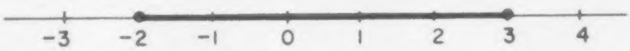
(e) $|8 + 5|$ _____ $|8| + |5|$

4. If $a < b$, where a and b are real numbers, write a true sentence expressing the order of $-a$ and $-b$.

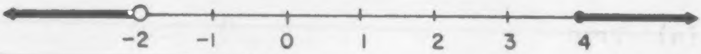
5. If $a < b$, is it possible to tell whether $-a < b$, $-a = b$, or $-a > b$? Give illustrations to support your answer.

6. Write an open sentence whose truth set is

(a)



(b)



7. If b is a negative number, indicate which of the following numbers are positive and which are negative.

(a) $-b$
(b) $|b|$
(c) $|-b|$
(d) $-|-b|$
(e) $-(-b)$
(f) $-|-|-b||$

8. Draw the graph of the truth set of each of the following open sentences.

(a) $|x| = 3$
(b) $|x| -1 = 5$
(c) $|x| = 0$
(d) $|x| < 0$
(e) $-|x| < 0$

9. Describe the truth set of each of the open sentences.

(a) $|x| > x$
(b) $|x| < x$
(c) $|x| = x$
(d) $|x| = -x$

10. Describe the variable and translate into an open sentence:

Peter lives one mile closer to school than Ralph. Peter is more than $3\frac{1}{2}$ miles from school. What distance is the school from Ralph's home?

11. Consider the set of real numbers
$W = \{-4, 3\frac{1}{2}, \pi, 0, -\frac{13}{4}, 1.42, 182, \sqrt{2}\}$.

Which elements of this set are

(a) integers?
(b) rational numbers but not integers?
(c) negative rational numbers?
(d) irrational numbers?
(e) non-negative real numbers?
(f) rational numbers that are greater than -4 and less than 2?

Answers to Suggested Test Items

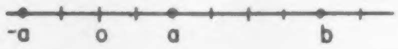
1. (a) True (d) True
   (b) False (e) False
   (c) True

2. $-\frac{5}{2}, -2, -\frac{1}{3}, -\frac{1}{8}, 0, \frac{1}{2}$

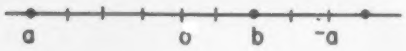
3. (a) $2 > -|-3|$ (d) $-\frac{14}{17} > -\frac{15}{17}$
   (b) $-4 > -7$ (e) $|8 + 5| = |8| + |5|$
   (c) $-|-3| = -3$

4. $-b < -a$

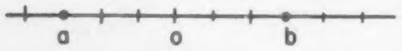
5. It is impossible to say whether $-a < b$, $-a = b$, or $-a > b$. The answer depends on the absolute values of a and b. The graphs below illustrate some possibilities.



$-a < b$



$-a > b$



$-a = b$

Numerical exercises such as the following can be used.

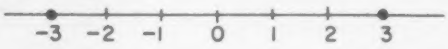
$-2 < 5$ and $-(-2) < 5$

$-7 < 5$ and $-(-7) > 5$

6. (a) $x \geq -2$ and $x \leq 3$
   (b) $x < -2$ or $x \geq 4$

7. (a) positive (d) negative
   (b) positive (e) negative
   (c) positive (f) negative

8. (a)



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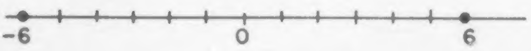

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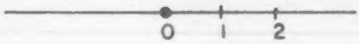
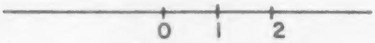
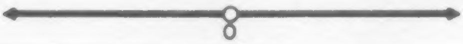


(d)



the empty set

(e)



all points except 0

9. (a) the set of negative real numbers

(b) $\emptyset$

(c) the set of non-negative real numbers

(d) the set of negative reals and zero

Note: $|0| = -0$

10. If Ralph lives x miles from school, then $x > 4\frac{1}{2}$.

11. (a) -4, 0, 182

(b) $3\frac{1}{2}$, $-\frac{13}{4}$, 1.42

(c) -4, $-\frac{13}{4}$

(d) π, $\sqrt{2}$

(e) $3\frac{1}{2}$, π, 0, 1.42, 182, $\sqrt{2}$

(f) 0, $-\frac{13}{4}$, 1.42

Chapter 7

ADDITION OF REAL NUMBERS

In this chapter we take up the study of addition. Our problem is essentially that of defining this operation on the larger set which includes the negatives. Though most students can achieve satisfactory competence in actual computations with these numbers through various intuitive devices, a formal definition is a necessary mathematical tool for the establishment of properties and a genuine understanding of the nature and structure of the real numbers.

We first consider some examples using gains and losses to suggest how addition involving negative numbers might be defined. The number line is also used to picture this. Finally, as an outgrowth of these experiments, a formal and precise definition is formulated.

The properties of addition are then presented, with stress on the fact that our definition of addition of real numbers permits the familiar properties of addition of the numbers of arithmetic to hold.

Very early in the chapter the student should learn how to find sums involving negative numbers. This is easy and is suggested completely by the profit and loss examples, and by the number line. However, our immediate objective is more ambitious that just teaching the arithmetic of negative numbers. We want to bring out the important fact that what is really involved here is an extension of the operation of addition from the numbers of arithmetic (where the operation is familiar) to all real numbers in such a way that the basic properties of addition are preserved. This means that we must define addition in terms of only the non-negative numbers and the familiar operations on them. The result in the language of algebra is a formula for $a + b$ involving the familiar operations of addition, subtraction, and taking opposites applied to the non-negative numbers, $|a|$ and $|b|$. The complete formula appears formidable because of the variety of cases. However the idea is simple and is nothing more than a general description of exactly what we always do in obtaining sums which involve one or more negative numbers.

The main problem is to lead up to the general definition of $a + b$ in a plausible way. We have chosen to make full use of the number line and especially to make use of absolute value.

7-1. Using the Real Numbers in Addition.

The profit and loss approach to addition of positive and negative numbers seems to be a natural one. The only thing which may seem new to the student is the representation in terms of positive and negative numbers.

Answers to Oral Exercises 7-1; pages 230-231:

1. (a) $5 + (-3) = 2$
   (b) $50 + ((-40) + (-25)) = -15$
   (c) $2 + (-5) = -3$
   (d) $(-6) + (-3) + (4) + (5) = 0$
   (e) $(-6) + (8) = 2$

2. (a) 9
   (b) -7
   (c) -3
   (d) 3
   (e) 3
   (f) -3
   (g) -13
   (h) 8
   (i) -6
   (j) 1
   (k) 1
   (l) -1

Answers to Problem Set 7-1; pages 231-232:

1. (a) 11
   (b) 4
   (c) (-12)
   (d) 5
   (e) 5
   (f) (-5)
   (g) (-5)
   (h) (-9)
   (i) 7
   (j) 2
   (k) -10.8
   (l) -5
   (m) (-1)
   (n) 1
   (o) (-2)
   (p) 1
   (q) 1
   (r) -2
   (s) 1

2. (a) $a = 3$
   (b) $a = 7$
   (c) $a = -7$
   (d) $b = -3$
   (e) $c = 6$
   (f) $m = -3$
   (g) $n = 0$
   (h) $n = 0$

pages 232-234: 7-1 and 7-2

(i) $m = -1$
(j) $c = 6$
(k) $b = -11$
(l) $a = -2\frac{1}{2}$
(m) $b = -.4$

7-2. Addition and the Number Line.

Recall that the main purpose of addition on the number line is to lead up to the definition of addition given on pages 239-240. By this time the students are familiar with the number line, and it is hoped that illustrating addition on it will seem natural. Note also that the concept of absolute value, introduced in the last chapter, is used extensively; it is central to the definition of addition developed here.

Although some of the exercises in this chapter, which are designed to strengthen understanding, will call for specific application of the formal definition, the students will not be expected to use this on all occasions as a rule by which to add real numbers. The point of view here is that the student now has a description of the process he has already learned how to do. In general a student should be encouraged to apply any intuitive process for addition of real numbers which he finds reliable.

Answers to Oral Exercises 7-2a; pages 234-235:

1. -5° F
2. 4° F
3. -3
4. 1° C
5. -1° C
6. (a) Start at zero. Move 5 units to the left, then 2 units to the right. The sum is -3.
   (b) Start at zero. Move 5 units to the left, then 2 more units to the left. The sum is -7.
   (c) Start at zero. Move 5 units to the right, then 2 more units to the right. The sum is 7.

pages 234-235: 7-2

(d) Start at zero. Move 5 units to the right, then 2 units to the left. The sum is 3.

(e) Start at zero. Move 6 units to the left, then 7 more units to the left. The sum is -13.

(f) Start at zero. Move 11 units to the left, then 15 units to the right. The sum is 4.

(g) Start at zero. Move 4 units to the right, then 12 more units to the right. The sum is 16.

(h) Start at zero. Move 6 units to the right, then 7 units to the left. The sum is -1.

(i) Start at zero. Move 6 units to the right, then 6 units to the left. The sum is 0.

(j) Start at zero. Move 7 units to the left, then no units either way. The sum is -7.

(k) Start at zero. Move 4 units to the right, then 6 units to the left. The sum is -2. Then move 8 more units to the left. The sum is -10.

(l) Start at zero. Move 5 units to the left, then 2 more units to the left. The sum is -7. Then move 7 more units to the left. The sum is -14.

(m) Start at zero. Move 4 units to the left, then 8 units to the right. The sum is 4. Then move 4 units to the left. The sum is 0.

(n) Start at zero. Move no unit in either direction, then move 2 units to the left. The sum is -2. Then move 2 units to the right. The sum is 0.

(o) Start at zero. Move 7 units to the right, then 2 units to the left. The sum is 5. Then move 3 units to the right. The sum is 8.

Answers to Problem Set 7-2a; pages 235-236:

1. (a) 7
   (b) 3
   (c) 3
   (d) -7
   (e) -3
   (f) -10
   (g) -4
   (h) 6
   (i) -2

pages 235-241: 7-2

2. (a) -5
(b) 3
(c) -6
(d) 6
(e) 0
(f) -2
(g) 3
(h) 0

3. (a) False
(b) False
(c) True
(d) True
(e) False
(f) True
(g) False
(h) True
(i) True
(j) True
(k) False
(l) False

4. (a) 4
(b) -1
(c) 8
(d) -6
(e) -8
(f) 6
(g) 8
(h) -6
(i) -8
(j) -14

5. (a) True
(b) False
(c) True
(d) True
(e) False
(f) True
(g) False
(h) True

6. (a) $\{-4\}$
(b) $\{6\}$
(c) $\{-2\}$
(d) $\{-2\}$
(e) $\{10\}$
(f) $\{8\}$
(g) $\{-7\}$
(h) $\{6\}$
(i) $\{-5\}$
(j) the set of numbers greater than 4
(k) the set of numbers greater than 0 (the positive numbers)
(l) the set of numbers less than (-2)
(m) the set of real numbers
(r) $\emptyset$

Answers to Problem Set 7-2b; page 241:

1. $(7) + (-3) = |7| - |-3|$
$= 7 - 3$
$= 4$

2. $(-7) + (-3) = -(|-7| + |-3|)$
$= -(7 + 3)$
$= -10$

3. $(-7) + 0 = -(|-7| + |0|)$
$= -(7 + 0)$
$= -7$

4. $3 + (-7) = -(|-7| - |3|)$
$= -(7 - 3)$
$= -4$

5. $7 + (-7) = 0$
since $|7| = |-7|$

6. $(-3) + 7 = |7| - |-3|$
$= 7 - 3$
$= 4$

7. $(-3) + (-7) = -(|-3| + |-7|)$
$= -(3 + 7)$
$= -10$

8. $0 + (-7) = -(|0| + |-7|)$
$= -(0 + 7)$
$= -7$

9. $(-3) + 3 = 0$
since $|-3| = |3|$

10. True
11. False
12. True
13. True
14. False
15. True
16. False
17. True
18. False
19. False

7-3. Addition Property of Zero; Addition Property of Opposites.

Note that the addition property of zero and the addition property of opposites are obtained directly from the definition of addition. Note also that the addition property of opposites says that the sum of a and $(-a)$ is zero. It does not say that if the sum of a and another number is zero, the other number is $(-a)$. This fact is proved later.

Answers to Oral Exercises 7-3; page 242:

1. True
2. False
3. True
4. True
5. False
6. True
7. True
8. False
9. True
10. False
11. False
12. False
13. True
14. True

Answers to Problem Set 7-3; pages 242-243:

1. 14
2. 0
3. 0
4. -8
5. -9
6. -45

7. 0
8. Any number greater than 3
9. Any number less than (-3)
10. Any number greater than 6
11. Any number less than (-4)

7-4. Properties of Addition.

The definition of the addition of real numbers has been made in terms of the non-negative numbers and the familiar operations upon them. We have seen that it agrees with our intuitive feeling for the operation of addition of real numbers as shown in working with gains or losses and with the number line. It is further required that addition of real numbers have the same basic properties that we observed for addition of numbers of arithmetic. It would be awkward, for instance, to have addition of numbers of arithmetic commutative and addition of real numbers not commutative.

Notice that, while we did not call them such for the students, the commutative and associative properties were, for all intents and purposes, regarded as axioms for the numbers of arithmetic, and the operation of addition was regarded essentially as an undefined operation. For the real numbers, however, we have made a definition of addition in terms of earlier concepts. If our definition has been properly chosen, we should find that the properties can be proved as theorems. While most students will not fully appreciate all this, the teacher should have it in mind as background.

We have tried to give the students a feeling for the provability of these properties, but very few of them will be ready to follow through the details. However, for the occasional student who is able and interested, we have left the way open for him to satisfy himself fully that the properties hold in all cases, not just in some particular cases he might try.

Answers to Oral Exercises 7-4; pages 244-245:

1. Yes
2. Commutative

3. (a) -2
   (b) -2
   (c) Yes
   (d) Associative

4. Two real numbers may be added in either order. The sum will be the same in both cases. The commutative property may also be more briefly stated: For any real numbers a and b, $a + b = b + a$.

5. For any real numbers a, b, and c, $(a + b) + c = a + (b + c)$. A precise word statement of this property becomes quite involved.

6. (a) Commutative
   (b) Associative
   (c) Associative
   (d) Commutative
   (e) Commutative and associative
   (f) Associative
   (g) Commutative
   (h) Commutative and associative
   (i) Commutative and associative
   (j) Commutative and associative

Answers to Problem Set 7-4; pages 245-246:

1. (a) $$\begin{aligned}(-3) + 7 + 5 + 3 + (-5) &= \big((-3) + 3\big) + \big(5 + (-5)\big) + 7 \\ &= 7\end{aligned}$$

   (b) $$\begin{aligned}14 + 6 + (-7) + 4 + 3 &= (14 + 6) + \big(-7 + (4 + 3)\big) \\ &= 20\end{aligned}$$

   (c) $$\begin{aligned}5 + (-8) + 6 + (-3) + 2 &= 5 + (-3) + (-8) + (6 + 2) \\ &= 5 + (-3) \\ &= 2\end{aligned}$$

   (d) $$\begin{aligned}(-9) + 5 + 6 + (-3) &= \big((-9) + 5\big) + \big(6 + (-3)\big) \\ &= -4 + 3 \\ &= -1\end{aligned}$$

   Here there is no particularly easy grouping. The student may want to add from left to right mentally, getting first (-4), then 2, and then (-1).

   (e) $$\begin{aligned}11 + (-17) + 9 + (-3) + 4 &= (11 + 9) + \big((-17) + (-3)\big) + 4 \\ &= 20 + (-20) + 4 \\ &= 4\end{aligned}$$

(f) $$\begin{aligned} c + 2 + (-c) + 5 &= \big(c + (-c)\big) + 2 + 5 \\ &= 0 + 7 \\ &= 7 \end{aligned}$$

(g) $$\begin{aligned} r + 4 + (-r) + (-4) &= \big(r + (-r)\big) + \big(4 + (-4)\big) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

(h) $$\begin{aligned} r + 6 + (-r) + (-3) &= \big(r + (-r)\big) + \big(6 + (-3)\big) \\ &= 0 + 3 \\ &= 3 \end{aligned}$$

2. (a) True
(b) True
(c) True
(d) False
(e) False
(f) False
(g) True
(h) False

3. (a) -6
(b) 7
(c) -9
(d) -3
(e) $2\frac{1}{2}$
(f) 8
(g) 10
(h) all real numbers
(i) all real numbers
(j) 4
(k) 18

7-5. Addition Property of Equality.

You may recognize the "Addition Property of Equality" as the traditional statement, "If equals are added to equals, the sums are equal." While we shall have frequent occasion to use this idea, we prefer not to treat it as a property of real numbers because it is really just an outgrowth of two names for the same number. The name "Addition Property of Equality" will be a convenient way to refer to this idea when we need to use it.

From another point of view, the addition property of equality can also be thought of as being a way of saying that the operation of addition is single valued; that is, the result of adding two given numbers is a single number. In other words, whenever we add two given numbers we always get the same result. Therefore, if a, b and c are real numbers and $a = b$, then the statement "$a + c = b + c$" can be thought of as saying that the result of adding the two given numbers was the same when they had the names "a" and "c" as when they had the names "b" and "c".

Though this property has more to do with the language we use in talking about numbers than with the numbers or the operations upon them, it is clearly a useful tool in finding the truth sets of sentences, and it will be put to use in this way in the next section of the text.

Answers to Oral Exercises 7-5; pages 248-249:

1. The resulting sentence is true.
2. The resulting sentence will not be true. The number represented by the right side will be larger than the number represented by the left side.
3. The resulting sentence will not be true. The same order relationship as in question 2 will exist.
4. (a) and (c) are statements of the property.
5. (a) all real numbers
   (b) all real numbers
   (c) all real numbers
6. (a) -5
   (b) 6
   (c) 11
   (d) -8
   (e) -9
   (f) 7
   (g) -1
   (h) 5
   (i) 12
   (j) 12
7. (a) -3
   (b) 4
   (c) 16
   (d) 8
   (e) 9
   (f) 3
   (g) -16
   (h) -16
   (i) -5
   (j) 14
   (k) 7
   (l) No number need be added, since by the use of the associative property of addition, the addition property of opposites, and the addition property of zero, the variable is isolated on the left side.

Answers to Problem Set 7-5; pages 249-250:

1. (a) True
   (b) True
   (c) False
   (d) True
   (e) True
   (f) False
   (g) True
   (h) True

2. In this problem we are not interested in the students' finding the truth set of an open sentence. The important thing here is for him to learn how to use the addition property of equality to obtain an equivalent open sentence. The student should not take short-cuts now.

(a) add (-4)
(b) add 6 and (-4), or 2
(c) add (-5)
(d) add 2
(e) add 2 and 7, or 9
(f) add (-6)
(g) add (-4) and 6, or 2
(h) add 30 and 10, or 40
(i) add 3(4 + 2), or 18
(j) No number need be added.

3. (a) 4
(b) 11
(c) -3
(d) -7
(e) $\frac{3}{2}$
(f) 1
(g) 11
(h) -8.4
(i) 9
(j) 15

7-6. Truth Sets of Open Sentences.

Later in Chapter 8 we shall learn about equivalent sentences and the permissible operations which keep sentences equivalent. For the present, however, notice that all we are claiming when we apply the addition property of equality is that if a number makes the original sentence true, it will make the new sentence true. We then have a chance to test each number of the truth set of the new sentence and see whether it makes the original sentence true. It is necessary to make this check every time, until we have the more complete reasoning of Chapter 8.

Attention is also focused on the fact that the use of the addition property of equality with a subsequent "checking" by substitution is more than a convenient alternative to guessing. It does, in fact, give us the complete truth set. In other words, the question concerning the possibility of additional truth numbers is definitely answered, in the negative. Students may have some difficulty in grasping this, but should be encouraged to try.

It is also necessary to reexamine at this point the general question of the domain of the variable since our basic set has been enlarged to include negative numbers. In Chapter 3 a covering statement was made to the effect that unless otherwise specified the domain of the variable was to be assumed to be all numbers of arithmetic for which the given sentence had meaning. A similar statement is made in this section. Since we do not wish to labor the point at this time as far as the student is concerned, no further discussion of domain is presented in the text. However, the teacher should be aware that until multiplication is defined for negative numbers in Chapter 8, such expressions as $3x$ or $5x$ are, theoretically, without meaning. Hence, in constructing the exercises and examples care has been taken to avoid attaching a coefficient to the variable for any sentence having a negative truth number.

Answers to Oral Exercises 7-6; page 254:

1. $(-\frac{2}{3})$
2. $(-\frac{1}{6})$
3. 9
4. (-3)
5. (-6)
6. 1.5
7. $3 + 2$, or 5
8. (-3)
9. 2
10. (-2)

Answers to Problem Set 7-6; pages 254-255:

1. The form of the student's answer, if he does not guess the truth number directly, is suggested in the examples in this section of the text. Parts (a) and (h) are written out in this manner below.

   (a) If $x + 5 = -3$ is true for some x,
   then $(x + 5) + (-5) = (-3) + (-5)$ is true for the same x

$$x + \big(5 + (-5)\big) = (-3) + (-5)$$
$$x + 0 = (-3) + (-5)$$
$$x = (-3) + (-5)$$

   and $x = -8$ is true for the same x.

   $\{-8\}$ is the truth set of the last sentence and since $(-8) + 5 = -3$ is true, $\{-8\}$ is the truth set for $x + 5 = -3$.

(b) $\{2\}$

(c) $\{-11\}$

(d) $\{-15\}$

(e) $\{0\}$

(f) $\{\frac{4}{3}\}$

(g) $\{-1.50\}$

(h) If $(-5) + 3x + (-8) = 15 + (-20) + 1$ is true for some x,

then $\Big((-5) + 3x\Big) + (-8) = \Big(15 + (-20)\Big) + 1$ is true for the same x

$$\Big(3x + (-5)\Big) + (-8) = \Big(15 + (-20)\Big) + 1$$

$$3x + \Big((-5) + (-8)\Big) = \Big(15 + (-20)\Big) + 1$$

$$\Big(3x + (-13)\Big) = (-5) + 1$$

$$3x + (-13) + 13 = (-4) + 13$$

$$3x + (-13 + 13) = 9$$

$$3x + 0 = 9$$

and $3x = 9$ is true for the same x.

$\{3\}$ is the truth set of $3x = 9$,

and since $(-5) + 3(3) + (-8) = 15 + (-20) + 1$ is true,

$\{3\}$ is the truth set of $(-5) + 3x + (-8) = 15 + (-20) + 1$.

2. (a) 4

(b) 6

(c) -1

(d) 5

(e) -5

(f) $-1\frac{1}{2}$

(g) 3.1

(h) set of all real numbers

3. (a) If $2x + (-5) = -3$ is true for some x,

then $\Big(2x + (-5)\Big) + 5 = -3 + 5$ is true for the same x

$$2x + \Big((-5) + 5\Big) = -3 + 5$$

$$2x + 0 = 2$$

$$2x = 2$$

1 is the truth number of $2x = 2$,

and since $2(1) + (-5) = -3$ is true,

1 is the truth number of $2x + (-5) = -3$.

(b) 5
(c) 2
(d) 1
(e) 5
(f) 3
(g) $\frac{5}{2}$
(h) $\frac{2}{3}$

7-7. Additive Inverse.

At the end of this section the student may be having his first experience at anything approaching a formal proof. His chief difficulty here is seeing the need for such a proof. We ask the student to extract from his experience the fact that for every number there is another number such that their sum is zero. At the same time the student can equally well extract from his experience that there is only one such number. Why then, do we accept the first idea from experience but prove the second? The reason is that we can prove the second. The two ideas differ in that one must be extracted from experience while the other need not be. The existence of the additive inverse is in this sense a more basic idea than the idea that there is only one such number. Speaking more formally, the existence of the additive inverse is an assumption; the uniqueness of the additive inverse is a theorem. You are referred to Haag, Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 2, Section 3, for further reading.

At this point we are still quite informal about proofs and try to lead into this kind of thinking gradually and carefully. The viewpoint about proofs in this course is not that we are trying to prove rigorously everything we say - we cannot at this stage - but that we are trying to give the students a little experience, within their ability, with the kind of thinking we call "proof". Don't frighten them by making a big issue of it, and don't be discouraged if some students do not immediately get the point. Discuss the proofs with them as clearly and simply as you can. We hope that by the end of the year they will have some feeling for deductive reasoning, a better idea of the nature of mathematics, and perhaps a greater interest in algebra because of the bearing of proof on the structure. For background reading on proofs the teacher is again referred to Haag, Studies in

Mathematics, Volume III, Structure of Elementary Algebra, Chapter 2, Section 3.

The principal emphasis in this section has been on introducing the student to formal proof. Notice, however, that the theorem through which this first experience in proof was given is itself a significant structural property of the real numbers.

Answers to Oral Exercises 7-7a; pages 256-257:

1. 5
2. -6
3. 7
4. -10
5. (a) -5
   (b) $x + 5 + (-5) = 0 + (-5)$
   (c) $\{-5\}$
6. Each is the additive inverse of the other.
7. (a) -4
   (b) 9
   (c) -25
   (d) 12
   (e) $-x$
   (f) x
   (g) $-(3m)$
   (h) $5k$
   (i) Either -8 or $-(3 + 5)$ or $(-3) + (-5)$
   (j) $-(x + 5)$
   (k) $-\big((-6) + 3m\big)$
   (l) $-\big((-4) + 2y + 5\big)$
   (m) $-(a + b)$
   (n) $-(3m + n)$
   (o) $-(4y - x + 2)$

Answers to Problem Set 7-7a; pages 257-259:

1. (a) $\{-2\}$
   (b) $\{-17\}$
   (c) $\{-7\}$
   (d) $\{-7\}$
   (e) $\{4\}$
   (f) $\{-14\}$
   (g) $\{17\}$
2. (a) $\{3\}$
   (b) $\{4\}$
   (c) $\{3\}$
   (d) $\{10\}$

(e) $\{3\}$
(f) $\{1\}$
(g) $\{2\}$

3. (a) $\{2\}$
(b) $\{5\}$
(c) $\{\frac{1}{2}\}$
(d) $\{2\}$
(e) $\{2\}$
(f) $\{3\}$
(g) $\{1\}$
(h) $\{-6\}$

4. (a) $\{0\}$
(b) $\{2\}$
(c) $\{0\}$
(d) $\{\frac{1}{7}\}$
(e) $\{\frac{1}{4}\}$
(f) $\{-1\}$
(g) If $3x + (-1) = 24 + (-4)$ is true for some x then $3x + (-1) + 1 = 24 + (-4) + 1$ is true for the same x.

$$3x = 21$$

$\{7\}$ is the truth set of $3x = 21$. Since $3(7) + (-1) = 24 + (-4)$ is true $\{7\}$ is also the truth set of the original open sentence.

(h) $\{1\}$

5. (a) $a + (-a) = 0$
(b) $x + (-5) = 15$
(c) $3x + (-\frac{1}{2}) = -x$
(d) $\left(x + (-3)\right)(x + 3) = x^2 - 9$
(e) $x - 5 = x + (-5)$
(f) yes, yes

6. If x is the number of nickels John had, then $4x$ is the number of pennies, and $2x$ is the number of dimes. The sentence is $5x + 4x + 20x + 7 = 94$.

7. If the smallest angle has n degrees, then the largest angle has $2n + 20$ degrees, and $n + (2n + 20) + 70 = 180$.

8. $x + x + 6x + 6x = 112$
where x is the number of inches in the width

Answers to Problem Set 7-7b; page 262:

1. Addition property of opposites

2. $\left(3 + (-4)\right) + \left((-3) + 4\right) = \left(3 + (-4)\right) + \left(4 + (-3)\right)$
   commutative property of addition

   $= \left(\left(3 + (-4)\right) + 4\right) + (-3)$
   associative property of addition

   $= \left(3 + \left((-4) + 4\right)\right) + (-3)$
   associative property of addition

   $= (3 + 0) + (-3)$
   addition property of opposites
   (or definition of addition)

   $= 3 + (-3)$
   addition property of zero

   $= 0$
   addition property of opposites

3. We can conclude that $-\left(3 + (-4)\right) = \left((-3) + 4\right)$ because of uniqueness of additive inverses.

4. The addition property of opposites

5. The addition property of opposites

6. The theorem on the uniqueness of the additive inverse of a real number

7. (a) $(-a) + (-3)$
   (b) $(-x) + (-y)$
   (c) $(-2m) + (-3)$
   (d) $(-3x) + (-2y)$
   (e) $2 + (-a)$
   (f) $(-x) + 3y$
   (g) $a + b$
   (h) $(-5x) + (-3)$

Answers to Review Problem Set; pages 264-268:

1. (a) 1
   (b) -4
   (c) 7
   (d) -4
   (e) 4
   (f) 6

2. (a) Since $|-5| > |3|$,
   $3 + (-5) = -(|5| - |3|) = -(5 - 3) = -2$

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(b) Since both are negative, $(-5) + (-11) = -(|-5| + |-11|)$
$= -(5 + 11) = -16$

(c) Since $|-15| > |0|$,
$0 + (-15) = -(|-15| - |0|) = -(15 - 0) = -15$

(d) Since $|\sqrt{2}| = |-\sqrt{2}|$,
$\sqrt{2} + (-\sqrt{2}) = 0$

(e) Since $|18| > |-14|$,
$18 + (-14) = (|18| - |-14|) = (18 - 14) = 4$

(f) Since $|\pi| = |-\pi|$,
$(-\pi) + \pi = 0$

(g) Since $|-\frac{2}{3}| > |\frac{1}{2}|$,

$$(-\frac{2}{3}) + \frac{1}{2} = -(|-\frac{2}{3}| - |\frac{1}{2}|) = -(\frac{2}{3}\cdot\frac{2}{2} - \frac{1}{2}\cdot\frac{3}{3})$$

$$= -(\frac{4}{6} - \frac{3}{6}) = -\frac{1}{6}$$

(h) Since both are negative,
$(-35) + (-65) = -(|-35| + |-65|) = -(35 + 65) = -100$

(i) Since 12 and 7 are numbers of arithmetic,
$12 + 7 = 19$

(j) Since $|10| > |-6|$,
$(-6) + 10 = (|10| - |-6|) = (10 - 6) = 4$

(k) Since $|\frac{-3}{2}| > |1|$,
$1 + (\frac{-3}{2}) = -(|-\frac{3}{2}| - |1|) = -(\frac{3}{2} - 1) = -\frac{1}{2}$

(l) Since $|-201| > |200|$,
$200 + (-201) = -(|-201| - |200|) = -(201 - 200) = -1$

3. (a) 7
(b) 3
(c) 5
(d) 13
(e) 28
(f) 9
(g) 105
(h) 6
(i) 30
(j) -10
(k) 1
(l) -6
(m) -6
(n) 7
(o) 12

4. (a) True
(b) True
(c) False
(d) True
(e) True
(f) False
(g) False
(h) False
(i) True

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5. (a) True
   (b) False
   (c) True
   (d) False for all x except $x = 0$
   (e) True
   (f) True

6. (a) 5
   (b) -3
   (c) -10
   (d) 2
   (e) -5
   (f) -8
   (g) 10
   (h) $\frac{1}{2}$
   (i) 0
   (j) -4
   (k) $-\frac{3}{2}$
   (l) 3
   (m) -1
   (n) 20
   (o) 1
   (p) 6
   (q) any number
   (r) no number
   (s) -11
   (t) 0

7. (a) $\{5\}$
   (b) $\{13\}$
   (c) $\{3\}$
   (d) $\{9\}$
   (e) $\{-1\}$
   (f) $\emptyset$
   (g) $\{4\}$
   (h) $\{-\frac{4}{3}\}$
   (i) $\{11\}$
   (j) $\{6\}$
   (k) $\emptyset$
   (l) $\emptyset$
   (m) $\{-5\}$

8. (a) associative property of addition and addition property of opposites
   (b) commutative property of addition
   (c) addition property of opposites and addition property of zero
   (d) associative property of addition and addition property of opposites and addition property of zero
   (e) associative property of addition
   (f) commutative property of addition
   (g) associative property of addition
   (h) commutative property of addition
   (i) addition property of opposites and addition property of zero
   (j) addition property of opposites, commutative property of addition, addition property of zero and associative property of addition.

9. The following are merely suggested methods. The student should use the properties in the way that makes the computation easiest for him. However, the teacher should discuss with the students the different ways of commuting and

associating the numbers.

(a) $\left((-\frac{1}{2}) + (-\frac{3}{2})\right) + 7 + \left((-2) + 2\right) = 5$

(b) $(\frac{5}{3} + \frac{1}{3}) + \left((-3) + (-2)\right) + 6 = 3$

(c) $\left(125 + (-25)\right) + \left((-17) + (-13)\right) = 70$

(d) $\left((-3) + (-5)\right) + 8 + (11 + 12) + \left((-4) + (-3)\right) = 16$

(e) $\left(\frac{2}{3} + (-\frac{5}{3})\right) + \left(\frac{3}{2} + (-\frac{1}{2})\right) + |-2| = 2$

(f) $\left(|-5| + (-5)\right) + 21 + (-8) + (-7) = 6$

(g) $(-9) + |-2| + 12 + |-7| + 7 = 19$

(h) $(-|-10|) + (-3) + (-|-6|) + \left((-15) + 15\right) = -19$

10. (a) If t is the number of feet above sea level that the tide registered, $t = (-0.6) + 5.1$.

(b) If b is the number of inches that Dave shot above the center of the target on the second shot, $b = 10 + (-3)$.

(c) If f is the number of feet that the submarine cruised below sea level after the change of position, $f = 254 + (-78)$.

(d) If x is the number of dollars that his daughter received, then $2x$ is the number of dollars that his son received, and $3x$ is the number of dollars that the widow received, so $x + 2x + 3x = 50{,}000$.

(e) If x is the total number of dollars that Mr. Johnson owed the bank, $x > 200$.

Suggested Test Items

1. Find a common name for each of the following:

(a)	$(-4) + (-11)$	(e)	$(-4) + 0$
(b)	$(-3) + 8$	(f)	$\left((-3) + \lvert -3 \rvert\right) + \lvert -5 \rvert$
(c)	$4 + (-6)$	(g)	$(-3) + \lvert (-3) + (-5) \rvert$
(d)	$(-5) + 5$	(h)	$x + (-x)$

2. Write a common name for each of the following, doing the addition in the easiest way. In each case tell what properties you used to make your work easier.

(a)	$(-17) + \left(30 + (-83)\right)$	(c)	$\left(23 + (-12)\right) + (-11)$
(b)	$\left((-19) + 183\right) + 19$	(d)	$\left((-98) + 102\right) + (-63)$

3. If a and b are two real numbers, determine whether the sum $a + b$ is positive, negative, or zero for each of the following cases.

(a) $a > 0, \quad b > 0$
(b) $a > 0, \quad b < 0,$ and $|a| > |b|$
(c) $a < 0, \quad b < 0$
(d) $a < 0, \quad b > 0,$ and $|a| > |b|$
(e) $a = -b$
(f) $a = 0, \quad b < 0$
(g) $a > 0, \quad b = 0$

4. Consider the sentences:

A. $3 + (-5) = (-5) + 3$
B. $(-8) + 8 = 0$
C. $\left(3 + (-\frac{5}{2})\right) + 7 = 3 + \left((-\frac{5}{2}) + 7\right)$
D. If $a = b$, then $a + 2 = b + 2$
E. $-(-4) = 4$
F. $0 + (-6) = -6$
G. $(-4) + (-3) = -(|-4| + |-3|)$

Which of the sentences illustrate:

(a) the commutative property of multiplication

(b) the addition property of zero

(c) the addition property of equality
(d) the fact that the sum of two negative numbers is the opposite of the sum of their absolute values
(e) the opposite of the opposite of a number is the number itself
(f) the associative property of addition
(g) the addition property of opposites
(h) the associative property of multiplication

5. Find the truth set of each of the following open sentences.

(a) $x + 2 = 7$
(b) $0 = 7 + n$
(c) $m + (-6) = 0$
(d) $(-6) + 7 = (-8) + a$
(e) $|x| + (-2) = 1$
(f) $x + |x| = 0$

6. When a certain number is added to 99, the result is 287.

(a) Write an open sentence to find the number
(b) Find the number by finding the truth set of the sentence

7. Which of the following sets of numbers is closed under addition?

(a) $\{-3, -2, -1, 0, 1, 2, 3\}$
(b) $\{\cdots, -12, -9, -6, -3, 0\}$
(c) the set of all negative real numbers

8. Describe in terms of operations with numbers of arithmetic.

(a) the sum of 8 and (-3)
(b) the sum of (-11) and 5
(c) the sum of (-12) and (-5)

9. Which of the following sentences are true? Which are false?

(a) $5 + (-5) < 0$
(b) $(-3) + 5 > 2 + (-7)$
(c) $(-9) + 3 < (-9) + 5$
(d) $|4 + (-6)| < |4| + |-6|$
(e) $(-\frac{1}{2}) + (-3) < -4$
(f) $24.9 + (-25.9) < |-1|$

10. (a) A number is three more than its additive inverse. What is the number? Find the answer to this question by finding the truth set of an open sentence. (Hint: If there is a number n such that $n = 3 + (-n)$, then $n + n = 3 + (-n) + n$ (why?), and $n + n = 3$ (why?).)

(b) A number is equal to its additive inverse. For what numbers is this sentence true? Answer this question by finding the truth set of an open sentence. (Hint: If there is a number n such that $n = -n$, then $n + n = (-n) + n$. Why?)

Answers to Suggested Test Items

1\.
(a) -15
(b) 5
(c) -2
(d) 0
(e) -4
(f) 5
(g) 5
(h) 0

2. (a)

$(-17)+\big(30+(-83)\big) = (-17)+\big((-83)+30\big)$ commutative property of addition

$= \big((-17)+(-83)\big)+30$ associative property of addition

$= (-100) + 30$

$= -70$

(b)

$\big((-19) + 183\big)+19 = \big(183 + (-19)\big)+19$ commutative property of addition

$= 183+\big((-19)+19\big)$ associative property of addition

$= 183 + 0$ addition property of opposites

$= 183$ addition property of zero

(c)

$\big(23 + (-12)\big)+(-11) = 23+\big((-12)+(-11)\big)$ associative property of addition

$= 23+(-23)$

$= 0$ addition property of opposites

(d) $\big((-98) + 102\big) + (-63) = 4 + (-63)$

$= -59$ There is no easiest way to do this.

3. (a) positive
(b) positive
(c) negative
(d) negative
(e) zero
(f) negative
(g) positive

4. (a) none
(b) F
(c) D
(d) G
(e) E
(f) C
(g) B
(h) none

5. (a) $\{5\}$
(b) $\{-7\}$
(c) $\{6\}$
(d) $\{9\}$
(e) $|x| + (-2) + 2 = 3$
$|x| = 3$ Truth set: $\{-3, 3\}$
(f) the set consisting of all negative real numbers and zero

For such numbers $|x| = -x$,
and so $x + |x| = x + (-x)$
$= 0$.

6. (a) $99 + n = 287$
(b) $n + 99 = 287$
$n + 99 + (-99) = 287 + (-99)$
$n = 188$ The truth set: $\{188\}$
The number is 188.

7. (a) not closed under addition
$(-3) + (-2) = -5$, and -5 is not an element of the set
(b) closed under addition
(c) closed under addition

8. (a) $8 + (-3) = |8| - |-3|$
$= 8 - 3$
$= 5$

(b) $(-11) + 5 = -(|-11| - |5|)$

$$= -(11 - 5)$$

$$= -6$$

(c) $(-12) + (-5) = -(|-12| + |-5|)$

$$= -(12 + 5)$$

$$= -17$$

9. (a) **False** (d) **True**
   (b) **True** (e) **False**
   (c) **True** (f) **True**

10. (a) If there is a number n such that

$$n = 3 + (-n),$$

$$\text{then } n + n = 3 + (-n) + n$$

$$2n = 3$$

$$n = \frac{3}{2}.$$

If $n = \frac{3}{2}$, then $-n = -\frac{3}{2}$, and "$\frac{3}{2} = 3 + (-\frac{3}{2})$" is true. Hence, $\frac{3}{2}$ is the required number.

(b) If there is a number n such that

$$n = -n,$$

then

$$n + n = (-n) + n$$

$$2n = 0$$

$$n = 0.$$

Since 0 is its own additive inverse, we have shown that 0 is the only number with this property.

Chapter 8

MULTIPLICATION OF REAL NUMBERS

This is the second of three chapters in which the operations with the numbers of arithmetic are extended to the real numbers and the properties of these operations are brought out. You may want to refer to the statement at the beginning of the commentary for Chapter 6 to have another look at the overall plan of these three chapters.

Background reading for the mathematics of this chapter is available in Studies in Mathematics, III, Chapter 3, Sections 2 and 4.

8-1. Products.

As in the case of addition, the point of view here is that we extend the operation of multiplication from the numbers of arithmetic to all real numbers so as to preserve the fundamental properties. This actually forces us to define multiplication in the way we do. In other words, it could not be done in any other way without giving up some of the properties.

The general definition of multiplication for real numbers is stated in terms of absolute values because $|a|$ and $|b|$ are numbers of arithmetic. The only problem for real numbers is to determine whether the product is positive or negative.

There are several ways of making multiplication of real numbers seem plausible. It seems best to let the choice of definition of multiplication be a necessary outgrowth of a desire to retain the distributive property for real numbers. At two points in Section 8-1, prior to the use of the distributive property to discover the nature of the products, there appear partial multiplication tables, included simply to help establish the plausibility of the definition of multiplication by permitting the student to see that the results obtained using the distributive property are the same as those seen in the extended multiplication table. Note, however, that if the definition of multiplication is based on considerations of mathematical structure, the implied extension of the symmetry of a multiplication table must be regarded only as supporting evidence for the definition.

Answers to Oral Exercises 8-1a; pages 270-271:

1. (a) 0
   (b) 0
   (c) 0
   (d) 0
   (e) 0
   (f) 0
   (g) 0
   (h) -5.2
   (i) 0
   (j) 0

2. (a) True
   (b) False
   (c) False
   (d) False
   (e) False

3. (a) true for all values of a
   (b) true for all values of a
   (c) not true for all values of n (in fact, true for no values of n)
   (d) true for all values of m
   (e) not true for all values of m (true for no values of m)
   (f) not true for all values of a (true for no values of a)
   (g) not true for all values of x and y (true for no values of x and y)

Answers to Oral Exercises 8-1b; page 274:

1. (a) 18
   (b) 0
   (c) 18
   (d) 0
   (e) -20
   (f) -24
   (g) -3
   (h) $-\frac{5}{12}$
   (i) -12
   (j) -3.6
   (k) -.32
   (l) 0
   (m) 0
   (n) -6
   (o) -5
   (p) 0
   (q) -5

2. The operations in (b) and (c) can be so performed, but not the operations in (a) and (d).

3. (a) -6
   (b) Yes
   (c) 6
   (d) "$(3)(-2)$" names the number -6.
   "$|3|\cdot|-2|$" names the number 6.
   Hence "$-(|3|\cdot|-2|)$" names the number -6.

4. (a) False
   (b) False
   (c) True
   (d) True

Answers to Problem Set 8-1b; page 275:

1. (a) True
   (b) False
   (c) True
   (d) False
   (e) True
   (f) True
   (g) False
   (h) False
   (i) True
   (j) False
   (k) False
   (l) True
   (m) True
   (n) False

2. (a) -5
   (b) -2
   (c) $-\frac{5}{8}$
   (d) -20
   (e) 0
   (f) -20

Answers to Oral Exercises 8-1c; page 278:

1. (a) 30
   (b) -12
   (c) -12
   (d) 12
   (e) 12
   (f) 0
   (g) 0
   (h) -7
   (i) 1
   (j) 0
   (k) $-3b$
   (l) $7b$
   (m) 0
   (n) 0

2. (a) 6
   (b) yes
   (c) 6
   (d) The expressions $|-2|\cdot|-3|$ and $(-2)(-3)$ both name 6.

3. (a) False
(b) True
(c) True
(d) True
(e) True
(f) True

Answers to Problem Set 8-1c; pages 279-281:

1. (a) 40
(b) (-24)
(c) 0
(d) (-40)
(e) (-24)
(f) (-9)
(g) 9
(h) 0
(i) 14
(j) 72
(k) 12
(l) 3
(m) 4
(n) 0
(o) $\frac{1}{12}$
(p) 4
(q) 3.5
(r) .18
(s) .8

2. (a) True
(b) False
(c) False
(d) True
(e) True
(f) True
(g) True
(h) False
(i) True
(j) False
(k) False
(l) True
(m) False
(n) True
(o) False
(p) True
(q) False
(r) False

3. (a) {-3}
(b) {-3}
(c) {-1,-2}
(d) the set of real numbers less than 6 and greater than -6
(e) the set of all real numbers except zero

4. One integer is n. The other integer is $n + 4$.
Their product is 5.
The open sentence is $n(n + 4) = 5$.
Possible pairs of integers whose product is 5 are
1 and 5 ; -5 and -1.
In either of the above cases, the second integer is 4 more than the first.
Therefore the integers are 1 and 5 or -5 and -1.

5. (a) 10
(b) 12
(c) 20
(d) 6
(e) 120
(f) 0
(g) 7
(h) 0

6. (a) True
(b) False
(c) True
(d) False
(e) False
(f) False
(g) True
(h) True

pages 280-284: 8-1

(i) True
(j) True
(k) False
(l) True
(m) False (Here the student should note that the left side of the equation is a positive number and the right side a negative number, and thus it is not necessary to simplify either side further to show that the sentence is false.)
(n) False
(o) True
(p) False
(q) True

7. (a) $\{-9, -6, -4, -3, -2, -1, 0, 1, 2, 3, 4, 6, 9\}$
(b) no
(c) yes
no

8. (a) $\{0, \frac{1}{4}, \frac{1}{2}, 1, 2, 4\}$
(b) no
(c) no. The product of any two negative numbers is positive.

9. (a) $\{\ldots, -\frac{9}{4}, -\frac{3}{2}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, \frac{9}{4}, \ldots\}$
(b) no

Answers to Oral Exercises 8-1d; page 284:

1. $(6)(-5) = -(|6| \cdot |-5|)$
2. $(-8)(-3) = |-8| \cdot |-3|$
3. $(2)(4) = |2| \cdot |4|$
4. $(5)(0) = |5| \cdot |0|$
5. $(-7)(1) = -(|-7| \cdot |1|)$
6. $(-\frac{2}{3})(2) = -(|-\frac{2}{3}| \cdot |2|)$
7. $(-5)(-\frac{1}{2}) = |-5| \cdot |-\frac{1}{2}|$
8. $(0)(-2) = |0| \cdot |-2|$
9. $(8)(-1) = -(|8| \quad |-1|)$
10. $(-2)(-3) = |-2| \cdot |-3|$
11. $(\frac{1}{3})(-\frac{1}{2}) = -(|\frac{1}{3}| \quad |-\frac{1}{2}|)$
12. $(-1)(-1) = |-1| \cdot |-1|$

pages 284-286: 8-1 and 8-2

13. $(0)(-\frac{7}{3}) = |0| \cdot |-\frac{7}{3}|$
14. $(8)(6) = |8| \cdot |6|$
15. $(-\frac{1}{4})(-\frac{1}{2}) = |-\frac{1}{4}| \cdot |-\frac{1}{2}|$
16. $(3)(-7) = -(|3| \cdot |-7|)$
17. $(-11)(2) = -(|-11| \cdot |2|)$
18. $(-5)(-\frac{2}{3}) = |-5| \cdot |-\frac{2}{3}|$
19. $(-6)(-6) = |-6| \cdot |-6|$
20. $(-7)(0) = |-7| \cdot |0|$

Answers to Problem Set 8-1d; page 285:

1. (a) If $a < 0$ and $b < 0$, then $ab = |a| \cdot |b|$.
   (b) If $a < 0$ and $b > 0$, then $ab = -(|a| \cdot |b|)$
   (c) If $a > 0$ and $b < 0$, then $ab = -(|a| \cdot |b|)$
   (d) If $a > 0$ and $b > 0$, then $ab = |a| \cdot |b|$.
   (e) If $a = 0$ and $b = 0$, then $ab = |a| \cdot |b|$.
   (f) If $a = 0$ and $b \neq 0$, then $ab = |a| \cdot |b|$.

2. (a) 45
   (b) -45
   (c) 45
   (d) 100
   (e) -1000
   (f) 10,000
   (g) -100,000
   (h) 8
   (i) -8
   (j) 8
   (k) -4.14
   (l) -4.14
   (m) -4.14
   *(n) 2
   *(o) 2

8-2. Properties of Multiplication.

Once the definition of multiplication of real numbers is formulated, it can be proved that the properties of multiplication which held for the numbers of arithmetic also hold for the entire set of real numbers.

The proof of the multiplication property of one is included in the student's text. Though it is probably the easiest property to prove among the properties dealt with in this section of the text, it will doubtless be difficult for slower students. The teacher should not expect mastery of the proof, but it is hoped that the proof can be followed by the student to the extent that it will provide an experience to give meaning to the assertion in his text that the properties of multiplication for the real numbers can be proved from our definition of multiplication.

pages 286-289: 8-2

The proofs of the associative property and the distributive property are not difficult, but are lengthy and tedious. The proof of the commutative property is quite brief and is given below:

If one or both the numbers a, b are zero, then $ab = ba$, by the multiplication property of zero. If a and b are both positive or both negative, then

$$ab = |a| \cdot |b|, \quad \text{and} \quad ba = |b| \cdot |a|.$$

Since $|a|$ and $|b|$ are numbers of arithmetic, and the commutative property holds for the multiplication of the numbers of arithmetic,

$$|a| \cdot |b| = |b| \cdot |a|.$$

Hence,

$$ab = ba$$

for these two cases.

If one of a and b is positive or 0 and the other is negative, then

$$ab = -(|a| \cdot |b|) \quad \text{and} \quad ba = -(|b| \cdot |a|).$$

Since

$$|a| \cdot |b| = |b| \cdot |a|,$$

and since if numbers are equal their opposites are equal,

$$-(|a| \cdot |b|) = -(|b| \cdot |a|).$$

Hence,

$$ab = ba$$

for this case also.

Since all possible cases have been considered, then

$$ab = ba, \quad \text{for any real numbers } a \text{ and } b.$$

Answers to Oral Exercises 8-2a; page 289:

1. (a) 4
   (b) -5
   (c) 12
   (d) -10
   (e) -9
   (f) -14

(g) 18
(h) -15
(i) 24
(j) 0
(k) -3
(l) 1
(m) 0
(n) -3b
(o) 1
(p) -6
(q) -6
(r) 8

__Answers to Problem Set 8-2a__; **pages 289-291:**

1. (a) commutative property of multiplication
   (b) commutative property of multiplication
   (c) associative property of multiplication
   (d) associative and commutative properties of multiplication
   (e) multiplication property of one
   (f) associative property of multiplication and multiplication property of one
   (g) associative property of multiplication and multiplication property of one
   (h) associative property of multiplication and commutative property of multiplication, or the associative property of multiplication and the multiplication property of zero
   (i) associative property of multiplication and the commutative property of multiplication, or the associative property of multiplication and the multiplication property of zero
   (j) distributive property
   (k) distributive property and commutative property of multiplication
   (l) commutative property of multiplication

2. (a) 2
   (b) 5
   (c) 4
   (d) -7
   (e) -7
   (f) -42
   (g) -24
   (h) -36
   (i) -24
   (j) 12
   (k) -18
   (l) 9
   (m) 1
   (n) 13
   (o) 0
   (p) 0
   (q) 18
   (r) 9
   (s) -6
   (t) 1
   (u) 0
   (v) (-2)x
   (w) 5a

3. (a) (2)(3) + (2)(-2)
   (b) (-3)(-4) + (-3)(-6)
   (c) 4(a) + 4(-5)
   (d) (1)(-5) + (1)(-10)
   (e) (1)(0) + (1)(-1)
   (f) (1)(-1) + (1)(1)

(g) $(5)\big((-6)a\big) + (5)(-2)$

(h) $(-6)(10x) + (-6)(-3)$

(i) $(6a)(4) + (-2)(4)$

(j) $(-8)(-7) + (-2)(-7)$

(k) $(1.2)(6m) + (-1.1)(6m)$

(l) $(-\frac{2}{3})\big((-\frac{2}{5})(2x)\big)$
$+ (\frac{3}{4})\big((-\frac{2}{5})(2x)\big)$

4. (a) $2(5 + 4)$

(b) $(-3)(7 + 3)$

(c) $4\big((-6) + (-9)\big)$

(d) $(-7)\big((-5) + (-4)\big)$

(e) $\big((-8) + (5)\big)y$

(f) $x\big((-9) + (-6)\big)$

(g) $1(4 + 6)$

(h) $1\big((-3) + 9\big)$

(i) $(-2)(a + b)$

(j) The distributive property is not useful to simplify this problem unless a and y name the same number.

Answers to Oral Exercises 8-2b; page 294:

1. -2
2. 3
3. $-a$
4. a
5. $-x + (-5)$
6. $x + (-5)$
7. $-x + 5$
8. $x + 5$
9. $-a + (-2)$
10. $-a + (-2)$
11. $-x + (-y)$
12. $-x + y$
13. $-xy$
14. $-xy$
15. xy
16. $-x + (-y)$
17. $-x + y$
18. $-a$
19. 0
20. $-2m + 4$

Answers to Problem Set 8-2b; pages 294-295:

1. (a) m check: $(-m) + (m) = 0$

(b) $m + (-4)$ check: $-m + 4 + m + (-4) = (-m) + m + 4 + (-4)$
$= 0$

(c) $2x + 3$ check: $-2x + (-3) + 2x + 3$
$= (-2x) + 2x + (-3) + 3 = 0$

(d) $-5y + (-7x)$ check: similar to above

(e) $y + x$ check: similar to above

(f) $3m + (-4)$ check: similar to above

(g) $(-1)(5x + \frac{3}{2})$ check: $5x + \frac{3}{2} + (-1)(5x + \frac{3}{2}) = 5x + \frac{3}{2} + (-5x) + (-\frac{3}{2})$

$$= 5x + (-5x) + \frac{3}{2} + (-\frac{3}{2})$$

$$= 0$$

(h) $8x^2 + 16x$ check: $\left(-8x^2 + (-16x)\right) + (8x^2 + 16x)$

$$= -8x^2 + 8x^2 + (-16x) + 16x = 0$$

(i) $(-3y^2) + 7y$ check: $3y^2 + (-7y) + (-3y^2) + 7y$

$$= 3y^2 + (-3y^2) + (-7y) + 7y = 0$$

(j) $-(2y + 11)$ check: $-\left(-(2y + 11)\right) + \left(-(2y + 11)\right)$

$$= 2y + 11 + (-2y) + (-11) = 0$$

2. (a) $8y$
   (b) $-m + (-2)$
   (c) $5x$
   (d) $7y$
   (e) $-m + 3$
   (f) 0
   (g) $-x$
   (h) $3y$
   (i) $-abk$
   (j) $3am$
   (k) $-4y^2$
   (l) $2mnx$

3. (a) $\{\frac{3}{2}\}$
   (b) $\{1\}$
   (c) $\emptyset$
   (d) the set of all real numbers

4. If x is the smaller of the numbers, the larger is $x + 5$, and

$$-\left(x + (x + 5)\right) = 17.$$

If $-\left(x + (x + 5)\right) = 17$ is true for some x,
then $-(2x + 5) = 17$ is true for the same x
$-2x + (-5) = 17$ is true for the same x
$-2x + (-5) + 5 = 17 + 5$ is true for the same x
$-2x = 22$ is true for the same x
$x = -11$ is true for the same x.

If x is -11, the left side of the first sentence is $-(-11 + \left((-11) + 5\right)$. This is 17, the same number as the right side, so {-11} is the truth set of the first sentence, and the two numbers required in the problem are -11 and -6.

5. Let d be the number of dollars spent by the daughter; then

$$d + (3d + 5) = 49.$$

The daughter spent $11, and the mother $33.

6. Let n be one number, then 3(-n) is the other number, and

$$n + 3(-n) = -86.$$

The numbers are 43 and -129.

8-3. Using the Multiplication Properties.

In this section, there is a series of "subsections", each of which introduces or emphasizes a particular kind of simplification or change in the form of a phrase. All of the processes are direct consequences of the properties of multiplication just developed. We wish to give sufficient practice with these techniques, but we wish also to keep them closely associated with the ideas on which they depend. We have to walk a narrow path between, on the one hand, becoming entirely mechanical and losing sight of the ideas and, on the other hand, dwelling on the ideas to the extent that the student becomes slow and clumsy in the algebraic manipulation. A good slogan to follow here is that manipulation must be based on understanding. We stress here again that the student must earn the right to "push symbols" (skipping steps, computing without giving reasons, etc.) by first mastering the ideas which lie behind and give meaning to the manipulation of the symbols.

In collecting terms we want the direct application of the distributive property to be the main thought. Don't give the impression that collecting terms is a new process. We are avoiding the phrases "like terms" and "similar terms" because they are unnecessary and tend to encourage manipulation without understanding.

Answers to Oral Exercises 8-3a; pages 297-298:

1. (a) $-12 + 4b$
   (b) $6 + 2b$
   (c) $-4m^2 + 5m$
   (d) $-am + (-cm)$
   (e) $28 + 8a$
   (f) 0

2. (a) $2(a + b)$
   (b) $(-5)(a + b)$
   (c) $3(3x + (-4)y)$
   (d) Either $\left((-4) + (-5)\right) m$, or $-9m$
   (e) $m(1 + m)$
   (f) The following are all correct:
   $(-2 + 4)x$, $2x$, $-2x\left(1 + (-2)\right)$

Answers to Problem Set 8-3a; pages 298-300:

1. (a) $15 + 5a$
   (b) $(-12) + 4b$
   (c) $6 + 2b$
   (d) $(-8) + (-4c)$
   (e) $(-3a) + (-3b)$
   (f) $(-2a) + 2b$
   (g) $5m + 5n$
   (h) $8.4a + (-4.8b)$
   (i) $-am + (-cm)$

2. (a) $2(a + b)$
   (b) $(-5)(a + m)$
   (c) $(-2)(b + c)$
   (d) $\left((-3) + (-4)\right)c$
   (e) $(-1)\left(m + (-n)\right)$
   (f) $\left((-2) + (-1)\right)m$
   (g) $a(b + m)$
   (h) $(x + r)y$
   (i) $m(r + 1)$
   (j) $(-\frac{2}{3})(m + n)$
   (k) $1.5(a + b)$
   (l) $2\left(m + (-2n)\right)$
   (m) $3(2a + 3b)$
   (n) $3\left(3x + (-4y)\right)$
   (o) $(-5)\left(2m + (3n)\right)$
   (p) $3\left(2a + (-5b)\right)$
   (q) $3a(3x + 4y)$
   (r) $(-2a)(2c + 3b)$
   (s) $m(1 + m)$
   (t) $a\left(a + (-2)\right)$
   (u) The distributive property does not apply.

page 299: 8-3

3. In some of the problems it is suggested that the teacher insist that the student follow the suggested steps in the solutions given for (k) and (l). Perhaps it would be helpful to ask them to identify the property that they have applied in each step.

(a) $(6 + 4)m = 10m$

(b) $(7 + (-3))a = 4a$

(c) $(9 + (-15))a = (-6)a$

(d) $((-5) + 14)y = 9y$

(e) $((-3) + (-6))m = (-9)m$

(f) $4a + 3b$. Call attention to the fact that since the distributive property does not apply here, the terms cannot be collected.

(g) $(4 + 1)a = 5a$

(h) $(-5)x + 2y$. The terms cannot be collected.

(i) $((-6) + 1)a = (-5a)$

(j) $((-1) + (-4))a = (-5a)$

(k) $2t + (-4)w + 3t + (-2)w$

$2t + 3t + (-4)w + (-2)w$ commutative property of addition

$(2 + 3)t + ((-4) + (-2))w$ distributive property

$5t + (-6)w$

(l) $(-5)a + (-2)b + 6a + 5b$

$(-5)a + 6a + (-2)b + 5b$ commutative property of addition

$((-5) + 6)a + ((-2) + 5)b$ distributive property

$a + 3b$

(m) $4b$

(n) $-x$

(o) $5a + (-3b)$

(p) $5m + (-3n)$

(q) $5a$

(r) This is already in its simplest form.

4. $$\begin{aligned} a(b + c + d) &= a\big(b + (c + d)\big) \\ &= a(b) + a(c + d) \\ &= ab + a(c) + a(d) \\ &= ab + ac + ad \end{aligned}$$

Be sure this problem is not overlooked. Though the student might do the next problem correctly without doing Problem 4, this problem shows him that it is the same familiar distributive property which justifies the work of simplification in exercises such as those in Problem 5.

5. (a) $2k(a) + 2k(m) + 2k(5)$
   (b) $(-6)a + (-6)(-b) + (-6)(-7)$
   (c) $\big((-3) + (-7) + 10\big)x$
   (d) $(-6)(c + b + a)$
   (e) $5a(a) + 5a(-2) + 5a(-a)$
   (f) $5a\big(a + 2 + (-1)\big)$; or, $5\big(a^2 + 2a + (-a)\big)$;
   or, $a\big(5a + 10 + (-5)\big)$

6. (a) $\emptyset$
   (b) $\{5\}$
   (c) the set of all real numbers
   (d) the set of all real numbers
   (e) the set of all real numbers
   (f) $\{-3\}$

7. Let x be the smallest of the numbers. Then

$$\begin{aligned} x + (x + 2) + (x + 4) &= 153 \\ x &= 49. \end{aligned}$$

The numbers are 49, 51, 53.

8. If n is the number of inches in the width,

$$2n + 2(n + 1) = 24$$

and the domain of n is the set of positive integers.

The truth set is $\{\frac{22}{4}\}$, but $\frac{22}{4}$ is not an integer.

Thus, it is not possible to find an integral length and width for this rectangle.

Answers to Oral Exercises 8-3b; page 302:

1. (a) $4m$
   (b) $-12xy$
   (c) $-(x)$
   (d) $-12a$
   (e) $8st$
   (f) $-28xy$
   (g) $15ab$
   (h) $-m$
   (i) $-4bc$
   (j) $3x + 2y$
   (k) $-6ac$
   (l) $5mn$
   (m) 0
   (n) 0
   (o) $-2abxy$
   (p) $12a^2m^2$

Answers to Problem Set 8-3b; page 302:

When the student has worked step by step through a number of exercises of this sort well enough to convince the teacher that he understands the process, then he certainly should be permitted to take short cuts in doing this work. The teacher should be ready, however, with occasional questions to be sure that the ideas behind the manipulation are always on call.

1. (a) $8a^2m$
   (b) $-12y^2$
   (c) $-6ab^2$
   (d) $24cd^2$
   (e) $-3c^2d$
   (f) $-6axy$
   (g) $4abcd$
   (h) $-12abxy$
   (i) $21am^2n^2$
   (j) a^2b^2cxy
   (k) $-9am^2x^2$
   (l) $72a^2m^2n^2$
   (m) $-m^2n^2$
   (n) $\frac{8}{15}\ a^2b^2c^2$
   (o) $4.5bc$
   (p) 0
   (q) $-35a^2 + (-2b)$

2. (a) $\{-1\}$
   (b) $\{1\}$
   (c) $\emptyset$
   (d) $\{0\}$

Answers to Problem Set 8-3c; page 303:

1. $6x^2 + 12xz$
2. $-18ax + 12bx$
3. $-2m^2 + 6mn$
4. $3mx + 3my$

5. $20am + 5cm$

6. $-a$

7. $ab + (-2)ac$

8. $cm + dm$

9. $b + (-c)$. This exercise and some of those which follow may also be done by use of the property that the opposite of the sum of two real numbers is the sum of their opposites.

10. $-4x + (-3y)$

11. $2x + (-3z)$

12. $4c + 6d$

13. $-4ac + (-6bc)$

14. $-10am + 15an$

15. $3m^2x + 4.5m^2y$

16. 0

17. $(-a) + (-b) + (-c)$

18. $(-b) + c + (-m)$

19. $-b + d + t$

20. $4mx + (-6my) + 8mz$

Answers to Problem Set 8-3d; page 304:

1. $a^2 + 5a + 6$

2. $a^2 + (-5a) + 6$

3. $a^2 + (-8a) + 15$

4. $b^2 + 10b + 24$

5. $c^2 + 2c + (-35)$

6. $m^2 + 7m + (-8)$

7. $m^2 + (-9m) + 20$

8. $m^2 + 2m + 1$

9. $t^2 + (-1)$

10. $x^2 + 6x + 9$

11. $a^2 + (-25)$

12. $x^2 + 8x + 15$

13. $x^2 + (-2x) + (-15)$

14. $k^2 + k + (-42)$

15. $k^2 + 2k + (-63)$

16. $b^2 + (-b) + (-2)$

17. $b^2 + 7b + (-8)$

18. $z^2 + (-7z) + 10$

19. $z^2 + (-10z) + 21$

20. $z^2 + (-2z) + 1$

21. $m^2 + (-6m) + 9$

22. $a^2 + 10a + 25$

23. $a^2 + (-10a) + 25$

24. $b^2 + 4b + 4$

25. $8 + 2b + (-b^2)$

26. $18 + (-9a) + a^2$

27. $36 + (-a^2)$

28. $8a^2 + 22a + 15$

29. $12m^2 + 10mn + 2n^2$

30. $ac + bc + ad + bd$

31. $x^2 + (-ax) + (-bx) + ab$

32. $x^2 + (-ax) + (-6a^2)$

pages 304-306: 8-3 and 8-4

33. $12a^2 + 19a + (-21)$

34. $2m^2 + (-11m) + 12$

35. $4m^2 + 4mn + n^2$

36. $4k^2 + (-b^2)$

37. $z^2 + (-5az) + 6a^2$

38. $9 + (-4z^2)$

39. $6x^2 + 13ax + 6a^2$

40. $4x^2 + (-10ax) + 6a^2$

8-4. Multiplicative Inverse.

By a series of examples and questions, and with the aid of the number line, the existence and uniqueness of the multiplicative inverse are set before the student.

The student's first opportunity to discover that zero has no multiplicative inverse comes in Oral Exercises 8-4a, Problem 2j. This point is emphasized again in the text in the section following these exercises. The student should understand not only *that* zero has no multiplicative inverse, but also *why* it does not.

The word "reciprocal" is not introduced until Chapter 10, where it is given as an alternative for multiplicative inverse. At that point the statement is made that the symbol $\frac{1}{x}$ is used to represent the reciprocal of the number x. Such a postponement is expedient since the student will not have encountered division with negative numbers in the present chapter. Hence, the symbol $\frac{1}{x}$ for $x < 0$ might cause trouble at this stage. In Chapter 10 the full connection between division and reciprocal can be established on a logical basis.

Some discussion should bring out the idea that the multiplicative inverse is unique, just as the additive inverse is unique. The uniqueness of the multiplicative inverse will be used in subsequent work.

Answers to Oral Exercises 8-4a; pages 305-306:

1. (a) 1
   (b) 1
   (c) 1
   (d) 1
   (e) 1
   (f) 1
   (g) 1
   (h) 1
   (i) 1

2. (a) $\{\frac{1}{4}\}$
   (b) $\{-\frac{1}{3}\}$
   (c) $\{-3\}$
   (d) $\{5\}$
   (e) $\{\frac{5}{4}\}$
   (f) $\{-\frac{7}{4}\}$
   (g) $\{\frac{2}{5}\}$
   (h) $\{-\frac{3}{4}\}$
   (i) $\{\frac{4}{9}\}$
   (j) $\emptyset$

Answers to Oral Exercises 8-4b; page 309:

1. 1
2. -1
3. Zero has no multiplicative inverse
4. There is no number n such that $n(0) = 1$.
5. yes
6. no
7. 1 and -1: $(1)(1) = 1$ and $(-1)(-1) = 1$
8. $-a$
9. a
10. no, zero does not have a multiplicative inverse.
    yes
11. The product of the two numbers will be one.

Answers to Problem Set 8-4b; pages 309-310:

1. (a) True
   (b) False
   (c) True
   (d) False
   (e) True
   (f) False
   (g) False
   (h) True
   (i) False
   (j) True
   (k) False
   (l) True
2. (a) $\{\frac{1}{3}\}$
   (b) $\{-\frac{1}{2}\}$
   (c) $\{5\}$
   (d) $\{\frac{1}{6}\}$
   (e) $\{-\frac{1}{12}\}$
   (f) $\{-\frac{1}{4}\}$
   (g) $\{\frac{1}{3}\}$
   (h) $\{-\frac{1}{2}\}$
   (i) $\{8\}$
   (j) $\{-3\}$
   (k) $\{\frac{1}{3}\}$

8-5. Multiplication Property of Equality.

Both the addition property of equality - comments for which the teacher may want to review at this point - and the multiplication property of equality are concerned with the language with which we work rather than the algebraic structure. If a, b, and c are real numbers and $a = b$, then the statement "$ac = bc$" can be thought of as saying that the result of multiplying two given numbers was the same when they had the names "a" and "c" as when they had the names "b" and "c".

As in the case of the addition property of equality, it is the usefulness of the multiplication property of equality in finding the truth sets of sentences that justifies its "elevation" to the status of a property.

Answers to Oral Exercises 8-5; page 312:

1. Using the multiplication property of equality, multiply each side of the sentence by $\frac{1}{2}$, since $\frac{1}{2}$, and 2 are inverses.
2. multiply each side by $\frac{1}{3}$
3. " " " " $\frac{1}{4}$
4. " " " " $\frac{1}{3}$
5. " " " " $-\frac{1}{5}$
6. " " " " $-\frac{1}{3}$
7. " " " " $-\frac{1}{2}$
8. " " " " $-\frac{1}{4}$
9. multiply each side by 2
10. " " " " -3
11. " " " " -2
12. " " " " -3
13. " " " " $-\frac{3}{2}$
14. " " " " $-\frac{6}{5}$
15. " " " " $-\frac{4}{3}$

Answers to Problem Set 8-5; page 312:

A few of the exercises in this set will be worked out in various degrees of detail. For the other exercises only the truth set will be given. In assigning exercises for students to work, it probably would be unwise to expect them to work out in detail more than four or five of these exercises. After all, it is also a worthwhile objective to get students to the point where they can determine the truth sets of open sentences of this type by inspection.

1. If there is an a such that

   $2a = 12$ is true,

   then the same a makes

   $\frac{1}{2}(2a) = 12(\frac{1}{2})$ and

   $(\frac{1}{2} \cdot 2)a = 12(\frac{1}{2})$ and

   $1 \cdot a = 6$ and

   $a = 6$ true.

2. $(-3)a = 15$

   $-\frac{1}{3}\left((-3)a\right) = (-\frac{1}{3})15$

   $\left(-\frac{1}{3}(-3)\right)a = (-\frac{1}{3})15$

   $1 \cdot a = -5$

   $a = -5$

3. $5a = -25$

   $\frac{1}{5}(5a) = (-25)\frac{1}{5}$

   $a = -5$

4. $\{2\}$
5. $\{0\}$
6. $\{-7\}$
7. $\{8\}$
9. $\{-9\}$
10. $\{10\}$
11. $\{6\}$
12. $\{-12\}$
13. $\{\frac{7}{4}\}$
14. $\{-\frac{17}{3}\}$
15. $\{\frac{5}{2}\}$
16. $\{-\frac{1}{4}\}$
17. $\{\frac{7}{18}\}$
18. $\{\frac{5}{6}\}$
19. $(-\frac{5}{6})c = \frac{10}{11}$

    $(-\frac{6}{5})\left((-\frac{5}{6})c\right) = (-\frac{6}{5})(\frac{10}{11})$

    $\left((-\frac{6}{5})(-\frac{5}{6})\right)c = (-\frac{6}{5})(\frac{10}{11})$

    $1 \cdot c = -\frac{60}{55}$

    $c = -\frac{12 \cdot 5}{11 \cdot 5} = -\frac{12}{11} \cdot \frac{5}{5}$

    $= -\frac{12}{11} \cdot 1$

    $c = -\frac{12}{11}$

20. $(-\frac{4}{3})b = (-\frac{9}{10})$

    $(-\frac{3}{4})(-\frac{4}{3}b) = (-\frac{3}{4})(-\frac{9}{10})$

    $b = -\frac{27}{40}$

8-6. Solutions of Open Sentences.

Equivalent sentences will be discussed in more detail in Chapter 15. You may wish to refer to this later discussion, in both text and commentary, before taking it up at this point. The idea is introduced here for linear equations because the student is probably beginning to be aware of it by now and surely is growing impatient with the checking routine. It is not our intention to do away with checking altogether for these equations, but rather to put it in its proper perspective - a check for errors in arithmetic.

It is important that the teacher note, and help the student note, that in the process of solving equations, not all steps involve directly the equivalence of two equations. Those steps in which the addition property and multiplication property of equality are used must raise the question of equivalence, but on the other hand there may be steps taken with the sole purpose of simplifying one member or both members of an equation.

Thus in going from

$$3x + 7 = x + 15$$

to $$(3x + 7) + \left((-x) + (-7)\right) = (x + 15) + \left((-x) + (-7)\right),$$

equivalence is an issue because, for example, the phrase on the left names a number different from that named by the left member of the original equation, as the addition property for equality has been used. But in going from

$$(3x + 7) + \left((-x) + (-7)\right) = (x + 15) + \left((-x) + (-7)\right)$$

to $$2x = 8,$$

the question of equivalence does not enter the picture because all that is happening is that each member of the equation is being written in simpler form. Both types of steps are important, of course, and students should be able to give reasons for them.

A prolonged discussion in the text of the difference between these steps could have been a distraction to the main idea, and so the task of emphasizing the distinction is largely the teacher's. This is probably appropriate, because many natural opportunities to point this out will arise in class discussion throughout the course.

In connection with the work on equivalent equations, some teachers report that classes have found good practice and enjoyment as well in the process of building complicated equations from simple ones by use of equivalent equations. For example,

pages 314-316: 8-6

$$\begin{aligned} x &= 3 \\ x + 1 &= 4 \\ 2(x + 1) &= 8 \end{aligned}$$

$$\begin{aligned} 2x + 2 &= 8 \\ 2x + 2 + 7 &= 8 + 7 \\ 2x + 9 &= 15 \end{aligned}$$

One of the principal reasons for introducing the idea of equivalent sentences at this time is the need for them in studying truth sets of inequalities, coming in Chapter 9. It is impossible, for example, to "check" the truth set of "$x + 8 > 10$" in the sense that one can check the truth set of "$x + 8 = 10$". It is important in the former case to know that "$x + 8 > 10$" and "$x > 2$" are *equivalent* sentences and so have *identical* truth sets. Therefore, no "checking" need be done in the original sentence (again, assuming no arithmetic errors); the truth set of "$x > 2$" *is* the truth set of "$x + 8 > 10$".

In the first example in this section of the text it is pointed out that the steps used in going from the original sentences to the simple sentence are reversible. Thus, if there is an x such that $2x + 5 = 27$ is true, then $x = 11$ is true for the same x; and, conversely, if there is an x such that $x = 11$ is true, then $2x + 5 = 27$ is true for the same x. Although it is not called by this name or stressed in the text, this is the first situation involving "if and only if", and it may be a good place for the teacher to begin building for this important concept, especially since the notion is perhaps more easily visualized in terms of equations and truth sets than in the more subtle proofs which the student may later encounter in other courses.

Although the idea is not difficult, "if and only if" often gives rise to confusion. The form always is "A if and only if B", where A and B are sentences. We are actually dealing with the compound sentence, "A *if* B and A *only if* B". The sentence "A if B" is a compact way to write "If B then A", and "A only if B" is a way of writing "If A then B". These conditional sentences are sometimes written "B implies A" and "A implies B". Some writers abbreviate "if and only if" to "iff". The compound sentence then reduces to "A iff B". The confusion with "if and only if" comes from trying to remember which statement is the "if" statement and which is the "only if" statement. Everyone has this trouble but it is fortunately not an important matter. What is important is that the compound sentence "A if and only if B" means "If A then B and if B then A".

The preceding remarks are for the benefit of the teacher only. It is probably not wise to introduce "if and only if" notation to the students at this time.

Answers to Problem Set 8-6; page 317:

Since the student has been shown procedures which assure the formation of equivalent sentences, it will no longer be necessary, in general, for him to "go the other way", i.e., carry out the reverse operations. In the first four problems, however, we give him this experience, which may, as suggested earlier, help set the stage for an understanding of "if and only if".

1.

$$\begin{aligned} 5x + (-4x) &= 7 \\ 5 + (-4)\,x &= 7 \\ 1x &= 7 \\ x &= 7 \end{aligned}$$

Going the other way:

$$\begin{aligned} x &= 7 \\ (1)x &= (1)(7) \\ 5 + (-4)\,x &= 7 \\ 5x + (-4x) &= 7 \end{aligned}$$

The truth set is $\{7\}$.

2. $\{2\}$
3. $\{2\}$
4. $\{3\}$
5. $\{\frac{15}{2}\}$
6. $\{3\}$
7. $\{-3\}$
8. $\{-\frac{5}{2}\}$
9. $\{-\frac{7}{3}\}$
10. $\{\frac{23}{7}\}$
11. $\{4\}$
12. $\{-\frac{17}{3}\}$
13. $\{-17\}$
14. $\{1\}$
15. $\{23\}$
16. $\{3\}$
17. $\emptyset$
18. $\{-\frac{7}{4}\}$
19. $\{-\frac{21}{4}\}$
20. any real number
21. $\{0\}$
22. $\emptyset$

8-7. Products and the Number Zero.

The theorem on products and the number zero is presented in this section in two parts. The first, since it is a direct consequence of the multiplication property of zero, requires very little elucidation. The second part is far less obvious. It is proved here in detail for two reasons: one, to dispel in the student's mind the erroneous notion that the first result implies the second, a common error; two, because of the significance of the second result in determining complete truth sets of certain types of equations. For example, without the second property we could not assert that 3 and 4 are the only truth numbers of the sentence $(x - 3)(x - 4) = 0$.

Our theorem can be stated in one piece as an "if and only if" statement as follows:

> For any real numbers x and y, $xy = 0$ if and only if $x = 0$ or $y = 0$. (The use of "or" here includes the case when both $x = 0$ and $y = 0$.)

As before, this form is not given in the text since the two part approach seems at this point to make for greater clarity.

Answers to Oral Exercises 8-7; page 320:

1. (a) True
(b) False
(c) True
(d) True
(e) False
(f) True
(g) False
(h) True

2. (a) {0}
(b) {0}
(c) {-1}
(d) {-3}
(e) {-1, -2}
(f) $\{-4, -\frac{1}{2}\}$

Answers to Problem Set 8-7; pages 320-323:

1. (a) {0}
(b) {0}
(c) {0}
(d) {0}
(e) {0}
(f) {0}
(g) the set of all real numbers
(h) {0}

2. (a) $\{4\}$
(b) $\{3\}$
(c) $\{7\}$
(d) $\{-2\}$
(e) $\{-6\}$
(f) $\{-1\}$
(g) $\{-\frac{2}{3}\}$
(h) $\{-.8\}$
(i) $\{1\}$
(j) $\{\frac{5}{4}\}$
(k) $\{-\frac{7}{8}\}$
(l) $\{\frac{5}{6}\}$
(m) $\{-\frac{3}{4}\}$
(n) $\{.91\}$

3. (a) $\{-4, -3\}$
(b) $\{-2, -6\}$
(c) $\{5, -6\}$
(d) $\{-10, 5\}$
(e) $\{3, 5\}$
(f) $\{-9, 9\}$
(g) $\{-\frac{2}{3}, \frac{1}{8}\}$
(h) $\{\frac{1}{2}, \frac{5}{6}\}$
(i) $\{\frac{3}{4}, -\frac{9}{12}\}$
(j) $\{-3.4, -2.18\}$
(k) $\{-5.15, 3.12\}$
(l) $\{1.75, 2.25\}$
(m) $\{0, -2\}$
(n) $\{0, 3\}$
(o) $\{0, -\frac{4}{5}\}$
(p) $\{0, \frac{7}{3}\}$
(q) $\{0, 1\}$
(r) $\{0, -1\}$
*(s) $\{-6, -\frac{5}{2}\}$
*(t) $\{-5, -8\}$
*(u) $\{4\}$
*(v) $\{\frac{3}{2}, -\frac{1}{2}\}$
*(w) $\{-\frac{8}{5}, -24\}$

4. Be sure that the students write out the steps carefully in the solutions. The following method is suggested:

(a) If x is the number of cents that Mr. Johnson paid for each foot of wire,
then $30x$ is the number of cents that Mr. Johnson paid for the first purchase of wire,
and $55x$ is the number of cents that Mr. Johnson paid for the later purchase of wire;

25x is the number of cents that the neighbor paid for the wire that he purchased.

Then the open sentence is

$$30x + 55x = 25x + 420$$

$$(30x + 55x) + (-25x) = 25x + 420 + (-25x)$$

$$30x + 55x + (-25x) = 25x + (-25x) + 420$$

$$\big(30 + 55 + (-25)\big)x = 0 + 420$$

$$60x = 420$$

$$\frac{1}{60}(60)x = (\frac{1}{60})(420)$$

$$x = 7$$

Checking: At 7¢ per foot,

30 feet of wire costs (30)(7) or 210¢;
55 feet of wire costs (55)(7) or 385¢;
Mr. Johnson's wire costs (210 + 385) or 595¢.
The neighbor's wire costs (25)(7) or 175¢ for 25 feet of wire.
Mr. Johnson's total cost is (175 + 420) or 595¢.
Thus 7¢ per foot is the cost of the wire.

(b) If n is the integer, $(n + 1)$ is the successor of that integer. The open sentence is

$$4n = 2(n + 1) + 10$$

$$4n = 2n + 2 + 10$$

$$4n = 2n + 12$$

$$\big(4n + (-2n)\big) = \big(2n + (-2n)\big) + 12$$

$$2n = 12$$

$$\frac{1}{2}(2n) = \frac{1}{2}(12)$$

$$n = 6$$

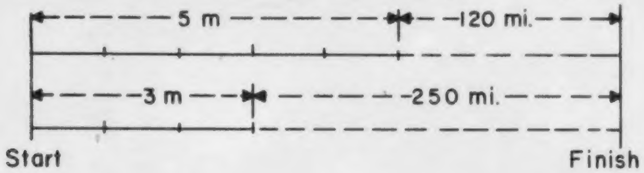
If the number is 6, four times the number is 24. If the number is 6, its successor is 7, twice the successor is 14, and 10 more than 14 is 24. Therefore 6 is the required integer.

(c) If m is the number of miles per hour that the first man drove,

then $5m$ is the number of miles that the first man drove in 5 hours

and $3m$ is the number of miles that the second man drove in 3 hours.

A diagram similar to the one below may be helpful.



The open sentence is

$$5m + 120 = 3m + 250.$$
$$5m + 120 + (-3m) + (-120) = 3m + 250 + (-3m) + (-120)$$
$$2m = 130$$
$$m = 65$$

Check: If each man drove at the rate of 65 miles per hour, then

the first man drove 325 miles in 5 hours,
the second man drove 195 miles in 3 hours;
$325 + 120 = 445$, $195 + 250 = 445$.

From here on the solutions are in more compact form and the check is not given.

(d) Let ℓ be the number of units in the length of the third side.

Open sentence:

$$\ell + (2\ell + 3) + (\ell + 5) = 44$$
$$4\ell + 8 = 44$$
$$4\ell = 36$$
$$\ell = 9$$

(e) n: the integer

$$n + (n + 1) = 1 + 2n$$
$$2n + 1 = 1 + 2n$$

The more alert students will observe that this sentence is true for any integer.

(f) a: the number of pigs

$$\begin{aligned} 4a + 2(a + 16) &= 74 \\ 6a + 32 &= 74 \\ 6a &= 42 \\ a &= 7 \end{aligned}$$

(g) b: the number of hits

$$\begin{aligned} 10b + (-5)(b + 10) &= -25 \\ 5b &= 25 \\ b &= 5 \end{aligned}$$

(Gain is 10 if he hits and -5 if he misses.)

Answers to Review Problem Set; pages 324-331:

1. (a) -3
   (b) 24
   (c) -25
   (d) -10
   (e) 0
   (f) 0

2. (a) 8
   (b) -16
   (c) 30
   (d) $-24m$
   (e) $24am$
   (f) $-5m^2$
   (g) $\frac{1}{2}my$
   (h) $\frac{8}{15}xy$
   (i) $24ab$
   (j) $-12bm$
   (k) $-45a^2$
   (l) 0
   (m) $-60a^2mx$

3. (a) $2a + 4b$
   (b) $-4c + 16d$
   (c) $42c + (-36d)$
   (d) $-32am + (-24an)$
   (e) $21ab + 28ac$
   (f) $16a^2 + (-2ab)$
   (g) $3a + (-6b)$
   (h) $4am + (-2an)$
   (i) $2bc + 3bd$
   (j) $5m^2 + (-10mn)$

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(k) $10c^2 + 20cd$

(l) $15bm + (-24b^2)$

(m) $-4ad + 4cd$

(n) $a^2 + 7a + 12$

(o) $a^2 + a + (-12)$

(p) $m^2 + (-36)$

(q) $y^2 + (-11y) + 18$

(r) $y^2 + \frac{5}{4}y + \frac{3}{8}$

(s) $z^2 + (-25)$

(t) $w^2 + (-\frac{3}{2}w) + \frac{9}{16}$

(u) $a^2 + 3a + 2.25$

(v) $b^2 + (-4.41)$

(w) $6y^2 + (-4yz) + (-8y)$

*(x) $-6y^2 + 9yz + (-4y) + (-12z) + 16$

*(y) $8m^2 + 2mn + 2m + n + (-1)$

4. (a) $13x$

(b) $-13a$

(c) $9k$

(d) $3b$

(e) n

(f) $9x$

(g) $-14a$

(h) $2a$

(i) $17p$

(j) 0

(k) $12a + 3c$

(l) $6a + 4b + c$

(m) $6p + 11q$

(n) $-2p + (-6r)$

(o) $-9b$

(p) 0

(q) $2t + 5s$

(r) $a + \frac{7}{6}b$

(s) $-4m + n + a$

(t) $7a$

5. (a) $\frac{31}{20}$

(b) $\frac{41}{24}$

(c) $\frac{21}{10}$

(d) $\frac{15}{16}$

(e) $\frac{27}{10}$

(f) $\frac{65}{48}$

6. (a) $3(a + b)$

(b) $-5(c + d)$

(c) $5(2m + n)$

(d) $-5(2a + 3b)$

(e) $n(m + ay)$

(f) $m\big(2y + (-x)\big)$;

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(g) $2b(x + 2y)$; or, $2(bx + 2by)$; or $b(2x + 4y)$.

In Problems (h) through (j) similar alternate answers are acceptable.

(h) $2a(2m + 3n)$

(i) $-3b(2x + 3w)$

(j) $\frac{2}{3}t(a + 2b)$

(k) $-\frac{5}{8}(b + c)$

(l) $2.5(m + 2n)$

7. (a) True
(b) False
(c) True
(d) False
(e) False
(f) False
(g) True
(h) False
(i) True

8. (a) True
(b) True
(c) False
(d) False
(e) True
(f) True
(g) False
(h) True
(i) False
(j) False
$a(-1) \neq 1$ is false when $a = -1$
(k) False when a is 0
(l) True

9. (a) False
(b) False
(c) True
(d) True
(e) False
(f) True
(g) False
(h) False
(i) False
(j) False
(k) True
(l) False
(m) True

10. (a) $\{2\}$
(b) $\{5\}$
(c) $\{-4\}$
(d) $\emptyset$
(e) $\{-9\}$
(f) $\{-8\}$
(g) $\emptyset$
(h) all real numbers
(i) $\{0\}$
(j) $\{7\}$
(k) $\{1\}$
(l) all real numbers greater than or equal to 5

(m) all numbers less than 1

(n) ∅

(o) all real numbers greater than 5 and all real numbers less than (-5)

(p) all real numbers

(q) {6}

(r) {-3}

(s) ∅

(t) {0}

(u) {1}

(v) all real numbers

(w) {-4}

(x) {0, 1}

(y) {-7, 2}

*(z) $\{0, -\frac{3}{2}, -3\}$

11. Students should be encouraged to check their answers. In verbal problems this checking should be done first in the original statement of the problem, then, if necessary, in the open sentence. Here the work is shown in detail only for parts (a) and (b). After a while the students should be able to omit some of the steps.

(a) Let x be the number.
Then the open sentence is

$$\begin{aligned} 2x + 5 &= 47 \\ 2x + 5 + (-5) &= 47 + (-5) \\ 2x &= 42 \\ x &= 21 \end{aligned}$$

Check: If 21 is the number, then twice the number is 42, and the sum of twice the number and 5 is 42 + 5 or 47.

(b) Let b be the number of bushels of wheat each truck can hold.
Then 3b is the number of bushels one truck hauled and 4b is the number of bushels the other truck hauled.

$$\begin{aligned} 3b + 4b &= 490 \\ (3 + 4)b &= 490 \\ 7b &= 490 \\ b &= 70 \end{aligned}$$

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Check: If each truck holds 70 bushels, then the first truck hauled 3(70) or 210 bushels, the second one hauled 4(70) or 280 bushels. Together they hauled 210 + 280 or 490 bushels.

(c) c: number of cents that one can of peaches costs

$$2c + 83 = 150 + (-4)$$
$$c = \frac{63}{2} \text{ or } 31\frac{1}{2}$$

(d) x: number of degrees in the second angle

$$x + 2x + (x + 12) = 180$$
$$x = 42$$
$$2x = 84$$
$$x + 12 = 54$$

Check: $42 + 84 + 54 = 180$

(e) Let t be the number of hours that the passenger train ran before overtaking the freight train. Then the freight train ran $(t + 1)$ hours.
$60t$ is the number of miles the passenger train traveled.
$40(t + 1)$ is the number of miles the freight train traveled.

$$60t = 40(t + 1)$$

(1) $t = 2$

$t + 1 = 3$

(2) 9:00 A.M.

(3) 120 miles

(f) Let w be the number of feet in the width.

$$w + w + (2w + 8) + (2w + 8) = 196$$
$$w = 30$$
$$2w + 8 = 68$$

The dimensions are 30 feet by 68 feet.

(If the student should say or write "w = 30 feet", remind him that w represents a number, so that

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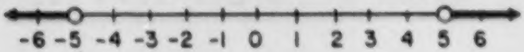
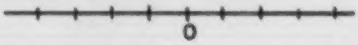
$w = 30$ is a statement about numbers, whereas "the width is 30 feet" is a statement indicating how long a certain line is.)

12. (a)

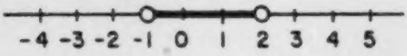

-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6

(b)

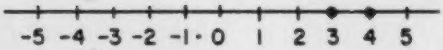


(the null set)

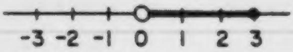
(c)


-4 -3 -2 -1 0 1 2 3 4 5

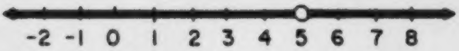
(d)


-5 -4 -3 -2 -1 0 1 2 3 4 5

(e)



(f)


-2 -1 0 1 2 3 4 5 6 7 8

Suggested Test Items

1. Find the value of each of the following when x is -3, y is 2, a is -4, and b is $\frac{1}{2}$.

   (a) $2ax + 3by$

   (b) $2ab\Big((-x) + y\Big)$

   (c) $\Big(a + x + (-y)\Big)^2$

   (d) $2x + (-a) + (-y^2)$

2. Write these indicated products as indicated sums.

   (a) $(-7)(3x + 4y)$

   (b) $3y\Big(y + (-2xy) + x^2\Big)$

   (c) $(x + 6)(x + 7)$

   (d) $\Big(5x + (-2)\Big)(3x + 7)$

   (e) $\Big(x + (-4)\Big)\cdot\Big(5x + (-3)\Big)$

   (f) $(\frac{1}{2}x)(-6xy)$

3. Write each of the following as an indicated product.

   (a) $7a + 7b$

   (b) $3m + 15n$

   (c) $4p + (-7px)$

   (d) $2xy + (-xy) + (-x)$

   (e) $(-4)a^2 + (-4)x^2$

4. Collect terms in the following.

(a) $z + 3z$

(b) $(-15a) + a$

(c) $4x + (-6y) + 6x + 12y$

(d) $x + 3y + 7x + (-2y) + 4y$

5. Write the multiplicative inverse of each of the following numbers.

(a) 3

(b) -5

(c) $\frac{16}{7}$

(d) $-3\frac{2}{3}$

(e) 0

(f) $|5 + (-7)|$

(g) $.23$

(h) $x + (-x)$

6. The following sentences are true for every a, every b, and every c.

A. $ab = ba$
B. $(ab)c = a(bc)$
C. $a(1) = a$
D. $a(0) = 0$
E. $(-a)(-b) = ab$
F. If $a = b$, then $ac = bc$.
G. $a(b + c) = ab + ac$

Which of the sentences expresses:

(a) the associative property of multiplication?
(b) the distributive property?
(c) the multiplication property of equality?
(d) the multiplication property of one?

7. Find the truth set of the following open sentences.

(a) $\frac{1}{3}x + (-8) = 4$

(b) $|-5| + 7 + (-5) + 2x = 0$

(c) $-\big((-5)x + 7\big) = 5x + (-7)$

(d) $|x| = 4(-3) + (-2)(-8)$

(e) $2x + 3x = 8 - 3x$

8. If a and b are real numbers, state the property used in each step of the following.

$$\begin{aligned}(a+b)\left(a+(-b)\right) &= (a+b)a + (a+b)(-b)\\ &= a^2 + ab + a(-b) + (-b^2)\\ &= a^2 + ab + (-ab) + (-b^2)\\ &= a^2 + 0 + (-b^2)\\ &= a^2 + (-b^2)\end{aligned}$$

9. Find truth sets for the following open sentences and draw their graphs.

(a) $7r + 4 + 3r = (-4r) + 18$

(b) $4(y + 2) + (-6)(y + 3) + (-y) = -4$

(c) $4|x| = 18 + (-2|x|)$

(d) $3\left(x + (-4)\right)\left(x + (-1)\right) = 0$

(e) $x(x + 2) = 0$

10. Write an open sentence for each of the following problems. State the truth sets and answer the questions.

(a) Two automobiles 360 miles apart start toward each other at the same time and meet in 6 hours. If the rate of the first car is twice that of the second car, what is the rate of each?

(b) Four times a certain integer is two more than three times its successor. What is the integer?

(c) The perimeter of a triangle is 40 inches. The second side is 3 inches more than the first side, and the third side is one inch more than twice the first side. Find the length of each side.

11. Which of the following sentences are true for all values of the variables? In each case tell what properties and definitions helped you decide.

(a) $a + (-a) = 0$

(b) $(73)(-41.3)(0) = 0$

(c) $(-\frac{1}{3})(\frac{17}{17}) = -\frac{1}{3}$

(d) $-(x + y) = -1(x + y)$

(e) $(-7)(-6\frac{1}{2}) > (7)(6\frac{1}{2})$

(f) $-5(n + 3) = -5n + (-5)(3)$

(g) $|-261| < (130)(-2)$ (h) $x + x = 2x$

12. Write and solve an open sentence to answer each of the following.

(a) When a number and twice its opposite are added, the result is $\frac{3}{2}$. For what number is this sentence true? Write and solve an open sentence to answer this question. Tell what properties you used in finding the solution of this sentence.

(b) Two numbers are multiplicative inverses, and one of them is one-fourth of the other. Find the pairs of inverses for which this sentence is true by writing and solving an open sentence.

(c) The product of a certain number and its opposite is the opposite of the square of the number. Find the number for which this is true by writing an open sentence and finding its truth set.

Answers to Suggested Test Items

1. (a) 27 (c) 81
(b) -20 (d) -6

2. (a) $(-21x) + (-28y)$ (d) $15x^2 + 29x + (-14)$
(b) $3y^2 + (-6xy^2) + 3x^2y$ (e) $5x^2 + (-23x) + 12$
(c) $x^2 + 13x + 42$ (f) $-3x^2y$

3. (a) $7(a + b)$ (d) $x\big(y + (-1)\big)$
(b) $3(m + 5n)$ (e) $-4(a^2 + x^2)$
(c) $p\big(4 + (-7x)\big)$

4. (a) $4z$ (c) $10x + 6y$
(b) $-14a$ (d) $8x + 5y$

5. (a) $\frac{1}{3}$ (c) $\frac{7}{16}$
(b) $-\frac{1}{5}$ (d) $-\frac{3}{11}$

(e) Zero has no multiplicative inverse.

(f) $\frac{1}{2}$

(g) $\frac{100}{23}$

(h) Zero has no multiplicative inverse.

6. (a) B

(b) G

(c) F

(d) C

7. (a) $\{36\}$

(b) $\{-\frac{7}{2}\}$

(c) the set of all real numbers

(d) $\{-4, 4\}$

(e) $\{1\}$

8. distributive property

distributive property

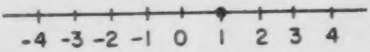
The product of one number and the opposite of another number is the opposite of the product of the two numbers.

addition property of opposites

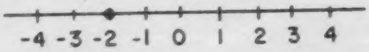
addition property of zero

(The associative property of addition is also used in the latter steps, since it makes possible the grouping implicit in these steps.)

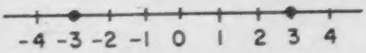
9. (a) $\{1\}$



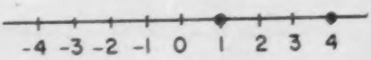
(b) $\{-2\}$



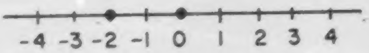
(c) $\{-3,3\}$



(d) $\{1, 4\}$



(e) $\{-2, 0\}$



10. (a) If the second car is traveling r miles per hour, the first car is traveling $2r$ miles per hour. In 6 hours, then, the second car travels $6r$ miles, and the first car $6(2r)$ miles. Since the number of miles traveled by both cars together is 360, we have

$$\begin{aligned} 6r + 12r &= 360. \\ 18r &= 360 \\ r &= 20 \\ 2r &= 40 \end{aligned}$$

The rate of the first car is 20 m.p.h., and the rate of the second car is 40 m.p.h.

(b) If n is the integer, $n + 1$ is its successor, and

$$\begin{aligned} 4n &= 3(n + 1) + 2 . \\ 4n &= 3n + 3 + 2 \\ 4n &= 3n + 5 \\ n &= 5 \end{aligned}$$

The integer is 5 and its successor is 6.

(c) If the first side is m inches long, the second side is $m + 3$ inches long, and the third side is $2m + 1$ inches long. Then

$$\begin{aligned} m + (m + 3) + (2m + 1) &= 40 . \\ 4m + 4 &= 40 \\ 4(m + 1) &= 4(10) \\ m + 1 &= 10 \\ m &= 9 \end{aligned}$$

The first side is 9 inches long, the second side is 12 inches long, and the third side is 19 inches long.

1. (a) True. Addition property of opposites

(b) True. Multiplication property of zero

(c) True. Multiplication property of one

(d) True. $-a = (-1)a$

(e) False. $(-a)(-b) = ab$

(f) True. Distributive property

(g) False. The absolute value of a number is always a non-negative number, and the product of a negative number and a positive number is a negative number.

(h) True. Distributive property

12. (a) Let x be the number, then -x is its opposite. The open sentence is

$$x + 2(-x) = \frac{3}{2}$$

$$x + (-2x) = \frac{3}{2} \qquad a(-b) = -ab$$

$$x\big(1 + (-2)\big) = \frac{3}{2} \qquad \text{Distributive property}$$

$$x(-1) = \frac{3}{2}$$

$$x = -\frac{3}{2} \qquad \text{Multiplication property of equality and } a(-1) = -a$$

The truth set is $\{-\frac{3}{2}\}$.

(b) Let x be one number and $\frac{1}{4}x$ be the other. The open sentence is

$$x(\tfrac{1}{4}x) = 1.$$

The truth set is $\{-2, 2\}$. The pairs of inverses are $-2, -\frac{1}{2}$ and $2, \frac{1}{2}$.

(c) Let x be the number, then -x is its opposite. The open sentence is

$$x(-x) = -x^2.$$

The truth set is the set of all real numbers.

Chapter 9

PROPERTIES OF ORDER

In this chapter the properties of the order relation "is less than" are systematically developed. Throughout the discussion all order relationships are phrased in terms of the symbol "<". The motivation for this is twofold, 1),to keep the development as uncluttered as possible, and 2), to emphasize the fact that we are essentially considering only one order relation among the real numbers. To be sure, in talking about a given pair of numbers, we may, and frequently do, shift from "less than" to "greater than" and back again without trouble. However, this tends to obscure the idea of order relation and is not permissible when we are studying the order relation "<" itself. We are making an issue of this matter because it is mathematically important for the student to begin thinking of order relation and not just order. It is not essential that they be able to explain it. If the teacher is careful to duscuss the properties correctly, then the student will automatically learn to think about an order relation as a mathematical object rather than as a convenient way of discussing a pair of real numbers.

It has been the custom in the past to assert that properties analogous to those applying to the order "is less than" can also be "proved" in a similar way for the order relation "is greater than". Rather than oppress the student with a host of additional properties, we make instead a simple statement to the effect that the expression $a < b$ may be written in an alternative form $b > a$, both expressions conveying precisely the same relationship idea, namely that the real number a is located to the left of the real number b on the number line.

The student has already been familiarized with the symbols of inequality and has used them in connection with open sentences. Thus, certain developments in this chapter may appear to be repetitive. It is hoped, however, that the student will be able to grasp the distinction between the use of the symbols "<" and ">" to form mathematical sentences and a study of the properties of an order relation. Again in this chapter we introduce some simple proofs. Considerable care should be taken to prepare the students for the presentation so that they might understand the significance of the proofs. The teacher may decide to omit some of the proofs. This can be done without loss of continuity. It is felt, however, that a strong effort should be made to present at least one or two of the proofs in class.

9-1. The Order Relation for Real Numbers.

It is likely that a student's unfavorable reaction to many of the properties presented in this chapter will stem from a sense that they are for the most part intuitively obvious. The comparison and transitive properties, for example, may seem scarcely worth mentioning. The teacher, however, should be aware of the fact that there are lumped together in our statement of the comparison property two distinguishable ideas: (1) a statement about the language of algebra and (2) a basic property of order. The first of these merely recognizes that it is possible for a and b to represent the same number. Then, of course, the order relation does not apply, since there is but one number involved. If, on the other hand, $a \neq b$, then exactly one of the following is true: $a < b$, or $b < a$. Some authors state the comparison property for $a \neq b$ only, thus stressing the order relation; others have termed the property the trichotomy property, thereby tending to stress the idea that, if a and b each represent any real number, it is always possible to "hang" exactly one of three symbols between them to make a true sentence. As indicated at the outset in our comments for this chapter, we hope in our approach to play down the tendency to focus on the numbers themselves, and to emphasize the order relation.

In connection with the transitive property it might be helpful to cite some examples, perhaps non-numerical ones, of a relation which does not have the transitive property. For instance, the fact that John is the father of Sam, and Sam is the father of Tom, does not imply that John is the father of Tom. Likewise if John loves Mary and Mary loves Joe, it will not always follow that John loves Joe! If the student is familiar with some elementary geometry, the relation "is perpendicular to" will provide a significant example of a non-transitive relationship between lines in the plane.

Answers to Oral Exercises 9-1; page 337:

In doing Problems 9 and 10, the student will very likely use the verb phrase "is greater than" in reading the sentence that is his answer. Thus the student will be more likely to clinch the idea that, given two different numbers, one is always less than the other.

1. $a < 4$
2. $-3 < d$
3. $x + (-2) < x + 2$
4. $y + (-3) < y + 1$
5. $-5 < -(m + (-1))$
6. $-6 < 3 + (-t)$
7. $m = 5$
8. $b < 5$, since $b < -1$ and $-1 < 5$
9. $-a > -4$; that is, $-4 < -a$
10. $0 > -c$; that is, $-c < 0$

Answers to Problem Set 9-1; pages 337-338:

1. (a) $-5 < -2$
   (b) $-\frac{3}{2} < -\frac{4}{3}$
   (c) $-5 < .01$
   (d) $\frac{5}{16} > .3124$; that is $.3124 < \frac{5}{16}$
   (e) $a > b$; that is $b < a$
   (f) $x < x + 1$
2. (a) $-a < -2$
   (b) $2 < b$
   (c) $-x < 3$
   (d) $y < 2$
   (e) $-|a| < 0$
   (f) $1 < x^2$
3. (a) $x + (-1) < 3$
   (b) $0 < z$
   (c) $2 < m$
   (d) $-(a + b) < b + (-a)$
   (e) $-|-3| < -2$ since $2 < |-3|$
   (f) Since $a = b$, there is but one number involved, and there can be no ordering.

9-2. Addition Property of Order.

The second addition property is introduced as an illustration of a simple deduction based on two other properties. Once again the result may seem "intuitively obvious".

Answers to Oral Exercises 9-2a; pages 340-342:

1. (a) True (b) True (c) True

(d) No decision can be reached. More information is needed.

(e) True

(f) True

(g) True

(h) True

(i) False

2. (a) The "=" relation does have the transitive property. If $8 - 5 = 3$ and $3 = 4 - 1$, then $8 - 5 = 4 - 1$.

(b) The relation ">" has the transitive property. If $7 > 3$ and $3 > -1$, then $7 > -1$.

(c) The relation $\neq$ does not have the transitive property. $8 \neq 7$, and $7 \neq 7 + 1$, but $8 = 7 + 1$.

This particular exercise provides a good opportunity for the teacher to point out that it requires only one, perhaps somewhat isolated counter-example to prove that a property does not hold. The student may easily be led to believe that the relation "$\neq$" is transitive since it would appear to work in all cases in which the original choice of c was such that $c \neq a$ to begin with. He may also be suspicious of the given answer on the ground that the hypotheses look like $a \neq b$ and $b \neq a$, with no c involved. Here again, it may be necessary to reaffirm the fact that different letters may be names for the same number.

(d) If a and b are any two different real numbers, then one of the statement, "$a < b$" and "$b < a$", is true and the other is false.

3. (a) $>$
(b) $<$
(c) $>$
(d) $<$
(e) $<$
(f) $>$
(g) $<$
(h) $<$
(i) $<$
*(j) $<$

Answers to Problem Set 9-2a; pages 342-343:

1. (a) $<$
(b) $<$
(c) $<$
(d) $>$
(e) No decision can be made.
More information is needed.

pages 343-346: 9-2

(f) $<$

(g) $<$ Since $a + 2 < -1$, $-1 < 0$, and $0 < b$

(h) Can't tell since c could be positive, 0, or negative.

(i) $<$ This problem anticipates the work at the close of Section 9-2.

(j) $<$

2. (a) False, $3 + 4 = \frac{12}{4} + 4$

(b) True, $-6 < -3$

(c) False, $3\frac{1}{8} < 5$

(d) True, $\frac{30}{32} < \frac{30}{31}$

(e) False, $3 + (-12) = (-18) + 9$

(f) True, $18 < 24$

(g) True, $(-273) < (-114)$

(h) False, $(-2\frac{1}{2}) < (-2)$

(i) True, $(-5.3) < 0.4$

(j) True, $(-\frac{15}{8}) = (-\frac{15}{8})$

__Pages 343-345__. Here and in Section 9-3 the concept of equivalent inequalities is presented without an attempt at rigorous justification. A more detailed treatment of the same topic is given in Chapter 15.

__Answers to Oral Exercises 9-2b__; pages 345-346:

1. (a) No (b) Yes (c) Yes

2. (a) Add (-3) to each "side" of the "$<$" statement.
$x < (-4) + (2) + (-3)$

(b) Add 8 to each side; $2n < (-27) + 8$

(c) Add (-4) to each side; $(-8) + 12 + (-4) < (-3n)$

(d) Add $(-\frac{8}{3})$ to each side; $7 + (-\frac{3}{2}) + 2 + (-\frac{8}{3}) < 2x$

(e) Add $\frac{1}{2}$ to each side; $.8 + 14 + (-\frac{2}{3}) + \frac{1}{2} < 4y$

3. (a) Add $-x + (-3)$ to each side, or add $-x$ to each side and (-3) to each side in two separate steps.

   (b) Add $4 + (-3y)$ to each side.

   (c) Combine terms to obtain $-4n + 14 < -3n$ then add $4n$ to each side.

   (d) Add $-\frac{4}{3} + \frac{y}{2}$ to each side.

   (e) Combine terms to obtain $.3 + 3.2y < .3 + 2.2y$ then add $-.3 + (-2.2y)$ to each side.

Answers to Problem Set 9-2b; pages 347-348:

1. (a) the set of all numbers less than 8

   (b) the set of all numbers less than 2

   (c) the set of all numbers less than 0

   (d) the set of all numbers greater than 0

   (e) the set of all numbers greater than $\frac{11}{10}$

   (f) the set of all numbers greater than $\frac{4}{3}$

   (g) the set of all numbers less than or equal to -4

   (h) the set of all numbers greater than or equal to $\frac{7}{3}$

   (i) $\emptyset$

   (j) the set of all numbers greater than $\frac{19}{2}$

   (k) the set of all numbers greater than 4

   (l) the set of all numbers greater than -7

   (m) the set of all numbers greater than -4

   (n) the set of all numbers greater than or equal to -1

   (o) $\{-3\}$

   (p) $\{1\}$

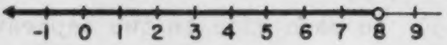
   (q) the set of all numbers less than 15

   (r) the set of all numbers

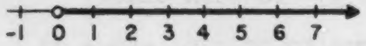
   (s) $\{-4\}$

   (t) the set of all numbers less than 5

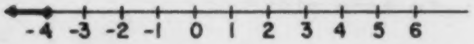
2. (a)



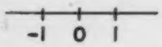
(d)



(g)

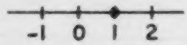


(i)

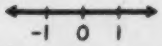


(the empty set)

(p)



(r)



3. (a) False (b) True (c) True

4. Suppose n is the number.

If $5n + 3 > 7 + 4n$,

then $5n + 3 + (-4n) + (-3) \geq 7 + 4n + (-4n) + (-3)$

and $n > 4$.

If $n > 4$,

then $n + 4n + 3 > 4 + 4n + 3$

$5n + 3 > 7 + 4n$.

All numbers greater than 4.

The teacher should note that we have asserted here that the "reversibility" of addition by a real number assures equivalent sentences. Therefore there is no need from the point of view of mathematical theory to reverse the steps in the process of solving the sentence. Nevertheless, going through the reverse steps does afford the student one means of checking his work for computational errors. He may prefer, instead, to choose several numbers from what appears to be the truth set of the sentence, and then check these in the original sentence, but this is an incomplete sort of check, hardly more than an indication of the plausibility of his supposed solution set.

5. Suppose x is the third test score.

$$\begin{array}{rl} \text{If } \dfrac{82 + 91 + x}{3} & > 90, \\ 173 + x & > 270, \\ x & > 97. \end{array} \qquad \begin{array}{rl} \text{Check: if } x & > 97, \\ 173 + x & > 270, \\ \dfrac{82 + 91 + x}{3} & > 90. \end{array}$$

He must score higher than 97.

Pages 348-350. The "two-way" connection between the order relation and addition will play a leading role in the development of the multiplication property. It is essential to the proof.

At this point the words equation and inequality are introduced as names for the two types of sentences under consideration. It is quite likely that these terms are already familiar to the students.

Answers to Oral Exercises 9-2c; page 350:

1. (a) $3 + 4 = 7$
   (b) $-2 + 6 = 4$
   (c) $-4 = -5 + 1$
   (d) $-\frac{9}{5} = -\frac{12}{5} + \frac{3}{5}$
   (e) $.99 + .009 = .999$
   (f) $-.3999 = -.4000 + .0001$
   (g) $(x) + 2 = x + 2$
   (h) $k + 1 = (k) + 1$

2. (a) $x < 6$
   (b) $w < 9.5$
   (c) $x < y$
   (d) $2x < y$
   (e) $m > n$
   (f) $x + (-1) < y + 2$ or $\big(x + (-1)\big) + 3 > y \quad (x = y)$
   (g) $x < y + 5$ or $x + 2 > y$
   (h) $k < 1$

Answers to Problem Set 9-2c; pages 350-351:

1. (a) $-15 > -24$; 9
   (b) $\frac{63}{4} > -\frac{5}{4}$; $\frac{68}{4}$
   (c) $\frac{6}{5} > \frac{7}{10}$; $\frac{1}{2}$

(d) $-\frac{1}{2} < \frac{1}{3}$; $\frac{5}{6}$

(e) $-254 > -345$; 91

(f) $-\frac{33}{13} < -\frac{98}{39}$; $\frac{1}{39}$

(g) $1.47 > -0.21$; 1.68

(h) $(-\frac{2}{3})(\frac{4}{5}) > (\frac{3}{2})(-\frac{5}{4})$; $\frac{161}{120}$

2. (a) True
   (b) False
   (c) True
   (d) True
   (e) True
   (f) True
   (g) False
   (h) True

3. Addition property of order
   Addition property of zero
   a + c names the same number as b.

9-3. Multiplication Property of Order.

A deductive argument is given to show the plausibility of the multiplication property of order. This argument does not constitute a complete proof of the property but it does contain the essential ideas that are involved in the proof.

A second multiplication property analogous to the second addition property is included. The results of this are fruitful in the study of square roots. They should be noted even though the student may wish to side-step the proof.

Answers to Oral Exercises 9-3a; page 357:

1. $2a < 10$
2. $-2b < 6$
3. $-p < 0$
4. $3m < 3n$
5. $-2q < 10$
6. $-3 < x + (-1)$
7. $15 < -3\big(a + (-b)\big)$
8. $25 < 5a + 5b$
9. $a < -4$
10. $-2 < x$

Answers to Problem Set 9-3a; pages 357-358:

1. (a) <
   (b) >
   (c) We can't say.
   (d) >
   (e) <
   (f) <

2. (a) $15 < -3x$
   (b) $a < -1$
   (c) $-2 < z + (-2)$
   (d) $-\frac{1}{5} < x$
   (e) $5 < x$
   (f) $y < z$

3. (a) $x = 6$
   (b) $z = -3$
   (c) $2x < 2$
   (d) $y < 1$
   (e) $m^2 + 1 > 1$
   (f) $t^2 < st$

4. (a) $25 < x^2$
   (b) $x^2 < 25$
   (c) $25 < z^2$
   (d) $z^2 < 25$
   (e) $a + b < 12$
   (f) $|b|^2 < |a|^2$

Answers to Oral Exercises 9-3b; page 360:

1. $\frac{1}{3}$
2. 3
3. -2
4. $-\frac{1}{3}$
5. $\frac{1}{4}$
6. $\frac{3}{2}$
7. $-\frac{8}{5}$
8. Do not multiply. Add -5.
9. Add -4 ; multiply by $\frac{1}{2}$
10. Add -3 ; multiply by $\frac{1}{3}$

Answers to Problem Set 9-3b; pages 360-361:

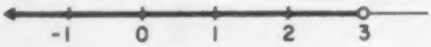
1. (a) all numbers less than 3
   (b) all numbers less than $\frac{3}{2}$
   (c) all numbers greater than $\frac{3}{16}$
   (d) all numbers less than 1
   (e) all numbers less than -3
   (f) all numbers $\frac{1}{7}$ or greater
   (g) all numbers greater than -18

(h) all numbers greater than -8

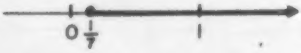
(i) $\{0\}$

(j) $\emptyset$

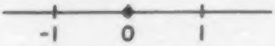
2. (a)



(f)



(i)



3. (a) all numbers less than (-3)

(b) all numbers greater than $(-\frac{11}{3})$

(c) all numbers $-\frac{9}{2}$ or greater

(d) $\emptyset$

(e) $\{\frac{11}{2}\}$

(f) all numbers greater than -12

(g) all numbers less than $.31$

(h) all numbers

(i) $\{\frac{1}{68}\}$

(j) all numbers greater than $-\frac{19}{11}$

4. If Moe pays x dollars, then Joe pays $(x + 130)$ dollars.

If $x + (x + 130) < 380,$	If $x < 125,$
$2x + 130 < 380,$	$2x < 250,$
$2x < 250,$	$2x + 130 < 380,$
$x < 125.$	$x + (x + 130) < 380.$

Joe: $x + 130 < 255,$
Moe: $x < 125;$
Total cost $< 380.$

Hence, Moe pays less than \$125.

5. Suppose n is the number.

$$\begin{array}{llll} \text{If} & 6n + 3 > 7 + 4n, & \text{If} & n > 2, \\ & 2n > 4, & & 2n > 4, \\ & n > 2. & & 6n + 3 > 7 + 4n. \end{array}$$

The number is greater than 2.

6. Suppose there are x students in the class.

$$\begin{array}{llll} \text{If} & 3x < x + 46, & \text{If} & x < 23, \\ & 2x < 46, & & 2x < 46, \\ & x < 23. & & 3x < x + 46. \end{array}$$

There are less than 23 students.

7. If Norma is x years old, then Bill is $(x + 5)$ years old.

$$\begin{array}{llll} \text{If} & (x + 5) + x < 23, & \text{If} & x < 9, \\ & 2x + 5 < 23, & & 2x < 18, \\ & 2x < 18, & & 2x + 5 < 23, \\ & x < 9. & & (x + 5) + x < 23. \end{array}$$

Norma is less than 9 years old.

Answers to Review Problem Set; pages 363-365:

1. (a) False
   (b) True
   (c) False
   (d) True
   (e) True
   (f) True
   (g) False
   (h) True
   (i) False
   (j) False

2. (a)

$$\begin{aligned} (-4) + 7 &< (-2)x + (-5), \\ 3 &< (-2x) + (-5), \\ 8 &< (-2x), \end{aligned}$$

and $-4 > x$

are all equivalent sentences.

Hence, the truth set is the set of numbers x such that $x < -4$.

page 363

(b) $4x + (-3) > 5 + (-2)x,$

$4x > 8 + (-2)x,$

$6x > 8,$

and $x > \frac{4}{3}$ are all equivalent sentences.

Hence, the truth set is the set of numbers x such that $x > \frac{4}{3}$.

(c) $(\frac{2}{3}) + (-\frac{5}{6}) < (-\frac{1}{6}) + (-3)x,$

$\frac{2}{3} + (-\frac{4}{6}) < (-3)x,$

$0 < (-3)x,$

and $0 > x$ are all equivalent sentences.

Hence, the truth set is the set of numbers x such that $x < 0$.

(d) $\frac{1}{2}x + (-2) < (-5) + \frac{5}{2}x,$

$3 < 2x,$

and $\frac{3}{2} < x$ are all equivalent sentences.

Hence, the truth set is the set of numbers x such that $x > \frac{3}{2}$.

(e) $\{-6\}$

(f) the set of all numbers $\frac{45}{2}$ or less

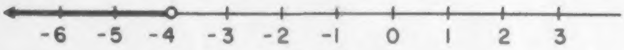
(g) $\{-\frac{7}{3}\}$

(h) all numbers less than $\frac{17}{6}$

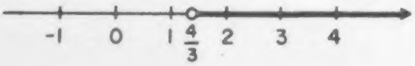
(i) all numbers greater than $-\frac{4}{5}$

(j) all numbers greater than 2

3. (a)



(b)



*4. If the rectangle is x inches wide, then it is $\frac{12}{x}$ inches long.

If $\frac{12}{x} < 6$ (where $x > 0$ because the number of inches in width is positive),

$12 < 6x$,

$2 < x$.

Hence, the width is greater than 2 inches.

*5. If the rectangle is x inches wide, then it is $\frac{12}{x}$ inches long.

If $4 < \frac{12}{x}$ and $\frac{12}{x} < 6$ (x is positive),

$4x < 12$ and $12 < 6x$,

$x < 3$ and $2 < x$.

Hence, the width is between 2 and 3 inches.

6. If $x > 0$, $x^2 > 0$.
If $x < 0$, $x^2 > 0$.
If $x \neq 0$, $x^2 > 0$.

If x is any non-zero number, $x^2 > 0$. If x is any real number, $x^2 \geq 0$.

7. (a) $3a^2 + (-6a^2b) + 3ab$

(b) $2x\left(2x + y + (-1)\right)$; or, $2\left(2x^2 + xy + (-x)\right)$;
or, $x\left(4x + 2y + (-2)\right)$

(c) $12a + 8a^2 + (-4ax)$

(d) $(x + 2)\left(a + (-4)\right)$

(e) $x^2 + x + (-2)$

(f) $6x^2 + 8x + 2$

(g) $-14y^2 + 32y + (-8)$

(h) $2m(2x + 1 + 3a)$; or $m(4x + 2 + 6a)$; or $2(2mx + m + 3am)$

(i) $6rt + (-12t) + 2r + (-4)$

(j) $x^2 + 4mx + 4m^2$

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8. (a) $-11x$

   (b) $7a + 2b$ can't be simplified further.

   (c) 0

   (d) $2rst + (-6stm)$ can't be simplified further.

   (e) $x + (-3y)$

9. If h is the number of hours required, then $34h$ and $45h$ are the distances traveled by the cars, giving the sentence

$$\begin{aligned} 45h &= 35 + 34h. \\ 45h + (-34h) &= 35 + 34h + (-34h) \\ h\big(45 + (-34)\big) &= 35 + 0 \\ 11h &= 35 \\ h &= \frac{35}{11} \end{aligned}$$

The time required was $3\frac{2}{11}$ hours.

10. n is the number of votes received by Charles.

$n + 30$ is the number of votes for Henry.

$$\begin{aligned} n + (n + 30) &= 516 \\ 2n + 30 &= 516 \\ 2n + 30 + (-30) &= 516 + (-30) \\ 2n &= 486 \\ n &= 243 \end{aligned}$$

Charles received 243 votes. (Note that the domain of the variable for this problem is the set of non-negative integers.)

11. a is the number of dollars left to the son.

$2a$ is the number of dollars left to the daughter.

$$\begin{aligned} a + 2a + 5000 &= 10{,}500. \\ 3a + 5000 + (-5000) &= 10{,}500 + (-5000) \\ 3a &= 5500 \\ a &= 1833.33\tfrac{1}{3} \end{aligned}$$

The son received \$1833.33.

12. (a) $-3a + 2b$

    (b) $2x + (-3a) + 7$

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(c) $-2a + 3 + (-2a) + (-3)$

(d) $3a + (-2b)$

(e) $-2x^2 + x + 1$

*13. (a) Prove that $-(a + b) = (-a) + (-b)$.

Proof:

$$\begin{aligned}(-a) + (-b) + (a + b) &= (-a) + a + (-b) + b \\ &\quad \text{commutative property of addition} \\ &= \big((-a) + a\big) + \big((-b) + b\big) \\ &\quad \text{associative property of addition} \\ &= 0 + 0 \\ &\quad \text{addition property of opposites} \\ &= 0 \\ &\quad \text{addition property of zero}\end{aligned}$$

This means that $(-a) + (-b)$ is the additive inverse of $a + b$. Therefore $(-a) + (-b)$ is equal to $-(a + b)$, since additive inverses are unique.

(b) Prove: if $a + c = b + c$ then $a = b$.

Proof:

$$\begin{aligned}a + c &= b + c \qquad \text{given} \\ a + c + (-c) &= b + c + (-c) \\ &\quad \text{addition property of equality} \\ a + \big(c + (-c)\big) &= b + \big(c + (-c)\big) \\ &\quad \text{associative property of addition} \\ a + 0 &= b + 0 \\ &\quad \text{addition property of opposites} \\ a &= b \\ &\quad \text{addition property of zero}\end{aligned}$$

With those students for whom Problem 13 is clearly too difficult as an independent exercise, the teacher may want to go through the proof in class, where students may be able to work profitably with it as a joint enterprise. This procedure is of considerable value to the teacher as well, for it gives him direct information regarding the degree of understanding and appreciation of formal proof that his students have at this stage of the course.

Suggested Test Items

1. We know that the sentence "$4 < 7$" is true. What true sentences result when both numbers are

   (a) increased by 5
   (b) changed by adding -5
   (c) multiplied by 5
   (d) multiplied by (-5)
   (e) multiplied by 0

2. Which of the following sentences are true ? Which are false?

   (a) If $a + 2 = b$, then $b < a$.
   (b) If $a + (-3) = b$, then $b < a$.
   (c) If $(a + 5) + (-2) = b$, then $b < a$.
   (d) If $a < 4$ and $4 > b$, then $a < b$.
   (e) If $a + 2 < 7$ and $b + 2 > 7$, then $a < b$.

3. Given $\frac{7}{9}$, $\frac{2}{3}$, $\frac{3}{4}$ and n. In each part of this problem make as many statements involving "$<$" about n and the given numbers as you can, if you know:

   (a) $n < \frac{7}{9}$
   (b) $n < \frac{2}{3}$
   (c) $n < \frac{3}{4}$

4. A man has three pieces of metal, each having the same volume. The sample of lead outweighs the sample of iron. The sample of gold outweighs the sample of lead. Which is a heavier piece of metal, gold or iron? What property of real numbers is illustrated here?

5. Find the truth sets of the following open sentences and draw their graphs.

   (a) $x + 5 < (-8) + |-8|$
   (b) $\frac{3}{2}x + (-3) > x + (-4)$
   (c) $(-4) + (-y) < \frac{3}{7} + (-\frac{3}{7})$
   (d) $37 + (-6r) + 7 > 9r + (-7r) + 8 + (-2r)$
   (e) $5n + (-3) > 2n + 9$
   (f) $4\,|x| > 12$

6. If p, q and t are real numbers and $p < q$, which of the following sentences are true?

   (a) $p + t < q + t$, if $t > 0$
   (b) $p + t > q + t$, if $t < 0$
   (c) $pt < qt$, if $t > 0$
   (d) $pt > qt$, if $t < 0$

7. Write an open sentence for each of the following problems. Find out all you can about the numbers asked for in the question.

   (a) Paul bought a jigsaw puzzle and put it together, only to discover that there were 13 pieces missing. If the label on the puzzle box said "over 350 pieces", how many pieces were in the puzzle when Paul bought it?

   (b) Tom has $12 more than Bill. After Tom spends $3 for meals, the two boys together have less than $60. How much money does Bill have?

   (c) If 13 is added to a number and the sum is multiplied by 2, the product is more than 76. What is the number?

   (d) Tom works at the rate of p dollars per day. After working 5 days he collects his pay and spends $6 of it. If he then has more than $20 left, what was his rate of pay?

   (e) A farmer discovered that less than 70% of a certain kind of seed grew into plants. If he has 245 plants, how many seeds did he plant?

8. If m is any positive real number and n is any negative real number, which of the following sentences are true?

   (a) $n < m$
   (b) $3n < 3m$
   (c) $2n < m + n$
   (d) $n + m < 2m$
   (e) $-n < -m$
   (f) $-m < -n$

Answers to Suggested Test Items

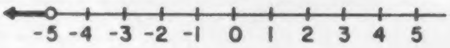
1. (a) $4 + 5 < 7 + 5$; or, $9 < 12$
   (b) $4 + (-5) < 7 + (-5)$; or, $-1 < 2$
   (c) $5(4) < 5(7)$; or, $20 < 35$
   (d) $-5(7) < -5(4)$; or, $-35 < -20$
   (e) $0(4) = 0(7)$; or, $0 = 0$

2. (a) False
   (b) True
   (c) False
   (d) False
   (e) True

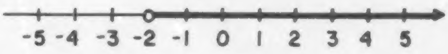
3. (a) No further statement
   (b) $n < \frac{7}{9}$; $n < \frac{3}{4}$
   (c) $n < \frac{7}{9}$

4. Since the number measuring the weight of iron is less than the number measuring the weight of lead, and the number measuring the weight of lead is less than the number measuring the weight of gold, by the transitive property of order, the number measuring the weight of iron is less than the number measuring the weight of gold. Hence gold is heavier than iron.

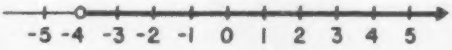
5. (a) The set of all real numbers less than -5



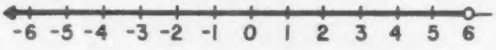
   (b) The set of all real numbers greater than -2



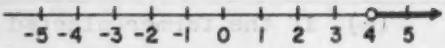
   (c) The set of all real numbers greater than -4



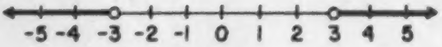
   (d) The set of all real numbers less than 6



(e) The set of all real numbers greater than 4



(f) The set of all real numbers less than -3 or greater than 3



6. (a) True (c) True
(b) False (d) True

7. (a) If Paul had p pieces in his puzzle when he bought it, then

$$p + 13 > 350.$$
$$p > 337$$

Thus there were more than 337 pieces left in Paul's puzzle.

(b) If Bill had B dollars, Tom had $B + 12$.

$$B + (B + 9) < 60.$$
$$2B + 9 < 60$$
$$2B < 51$$
$$B < 25.50$$

Thus Bill had less than $25.50.

(c) If n is the number required,

$$2(n + 13) > 76.$$
$$2n + 26 > 76$$
$$2n > 50$$
$$n > 25$$

The number is greater than 25.

(d) If Tom works at the rate of p dollars per day,

$$5p + (-6) > 20.$$
$$5p > 26$$
$$p > \frac{26}{5}$$

Tom's rate of pay is more than $5.20 per day.

(e) If the farmer planted p seeds,

$$245 < (.70)p.$$

$$\left(\frac{10}{7}\right)(245) < \left(\frac{10}{7}\right)\left(\frac{7}{10}\right)p$$

$$350 < p$$

The farmer planted more than 350 seeds.

8. (a) True
   (b) True
   (c) True
   (d) True
   (e) False
   (f) True

Chapter 10

SUBTRACTION AND DIVISION OF REAL NUMBERS

The logical structure of arithmetic and algebra could be developed without even mentioning subtraction or division. However, it is convenient to have the binary operations of subtraction and division, if only for ease in writing. Evidently, these operations must be defined directly in terms of the basic operations of addition and multiplication.

There are two eouivalent ways of defining subtraction either of which could have been used here. They are

(1) $a - b = a + (-b)$

(2) $a - b$ is the solution of the open sentence in x, $a = b + x$.

The writers of this book chose the first of these because it lends itself more readily to the point of view that subtraction of a number is a kind of inverse operation to addition of that number, an operation which is already known for numbers of arithmetic and must be extended to all real numbers. Thus we have only to identify subtraction in arithmetic with $a + (-b)$ in order to motivate the definition for all real numbers. This definition also builds on the work done previously with the additive inverse, which is important in its own right, and fits in nicely with the picture of addition and subtraction in the number line.

There are also two ways of defining division:

(1) $\frac{a}{b} = a \cdot \frac{1}{b}$

(2) $\frac{a}{b}$ is the solution of the equation $a = bx$, $b \neq 0$.

In this case also the first method was chosen because it parallels the chosen definition of subtraction and emphasizes the multiplicative inverse. It should also be mentioned that from these definitions the various properties of subtraction and division flow easily from earlier properties of addition and multiplication.

The second method of defining subtraction and division uses "solution of equations" as motivation. It has some advantage when the objective is to motivate extensions of the number system by demanding that certain simple equations always have solutions. For example the equation $a = b + x$ does not always have a solution in the positive integers (even if a and b are positive integers) but does always have a solution when the system is extended to include the negative integers. Similarly, the equation $a = bx$ $(b \neq 0)$ does not always have a solution in the integers (even if a and b are integers) but does always have a solution when the system is expanded to include the rational numbers. In later courses the introduction of the complex numbers is motivated by the demand that $x^2 + a = 0$ (in particular $x^2 + 1 = 0$) have a solution for every a.

The student is motivated by being asked to describe subtraction of numbers of arithmetic in terms of what must be added to the smaller to obtain the larger. When it is established that we must add the opposite of the smaller, we immediately take this as the definition of subtraction for all real numbers. A similar motivation leads to the definition of division.

Reference to subtraction and division will be found in Studies in Mathematics, Volume III, Section 3.1.

10-1. The Meaning of Subtraction.

We assume that the student is familiar in arithmetic with subtracting b from a by finding how much must be added to b to obtain a. From this our knowledge of equivalent equations quickly leads to adding the opposite of b to a.

For the student who has been subtracting by "taking away", we hope the illustration of making change will help the transition to an additive viewpoint.

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$$20 - 9 = 20 + (-9) = 11$$
$$10 - 15 = 10 + (-15) = -5$$
$$(-8) - 6 = (-8) + (-6) = -14$$
$$(-10) + (-7) = (-10) + 7 = -3$$
$$7 - (-4) = 7 + 4 = 11$$
$$(-5) - 2 = (-5) + (-2) = -7$$
$$5 - (-2) = 5 + 2 = 7$$

We read "$5 - (-2)$" as "five minus the opposite of 2". The first "-" indicates subtraction. The second "-" means "the opposite of". (Of course in this case the second could also be read "negative 2". If a variable were involved, however, the "-" would have to be read "the opposite".)

We shall soon want our students to be able to look at $a - b$ and think of it as a sum, the sum of a and $(-b)$. This is justified by our definition of subtraction.

You have, no doubt, noticed that we are not using the word "sign" for the symbol "-" or "+". We find that we do not really need the word, and since its misuse in the past has caused considerable lack of understanding (in such things as "getting the absolute value of a number by taking off its sign") we prefer not to use the word "sign" in any of our exposition.

A related point that we should mention is that we do not write $+5$ for the number five. The positive numbers are the numbers of arithmetic. We therefore do not need a new symbol for them. Thus we write 5, not $+5$, and the symbol "+" is used only to indicate addition.

Answers to Oral Exercises 10-1; page 371:

1. (a) $5 + (-4)$
   (b) $11 + (-12)$
   (c) $-4 + (-8)$
   (d) $-11 + 5$
   (e) $24 + 8$
   (f) $4a + (-3a)$
   (g) $-2x + 2$
   (h) $7y + 2y$
   (i) $\big(8 + (-12)\big) + (-2)$
   (j) $2 + \Big(-\big(8 + (-12)\big)\Big)$
   (k) $\pi + \pi$
   (l) $8k + 11k$
   (m) $6x + (-2x)$
   (n) $0 + 3m$
   (o) $6\sqrt{2} + (-9\sqrt{2})$

Answers to Problem Set 10-1; pages 371-373:

1. (a) $-(a + 7) = (-a) + (-7)$
   $= -a - 7$
   (b) $-(a - 7) = -a + \big(-(-7)\big)$
   $= -a + 7$
   (c) $-x^2 - x - 2$
   (d) $-x^2 + x + 2$
   (e) $x - y$
   (f) $x + y$

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2. (a) $15 + (-25) = -10$

(b) $132 + 18 = 150$

(c) $-12 + 24 = 12$

(d) $-7b + (-12b) = -19b$

(e) $-3x + 4x = x$

(f) $\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

(g) $7m + (-m) + (-12)$
$= 6m + (-12)$

(h) $-4x + (-2x) + b$
$= -6x + b$

(i) $\frac{3}{4}x + (-\frac{1}{5}x) = \frac{11}{20}x$

(j) $7.4m + (-12) + 3.5m$
$= 10.9m + (-12)$

3. (a) $2x + 2$
(b) $5 - w$
(c) 0
(d) x
(e) $1 - 2x$
(f) $2a - 4$
(g) -3
(h) $-x^2 + 2x$
(i) 0
(j) $-6m$
(k) $4m - 3$
(l) 2
(m) $2a + 2b$
(n) 18
(o) $\frac{1}{10}$
(p) $10 - 3a - 2b$
(q) 0
(r) 0

4. (a) $x + (-5) = -4$
$x + (-5) + 5 = -4 + 5$
$\{1\}$

(b) $\{-1\}$

(c) $\{\frac{11}{2}\}$

(d) $$4 + (-\frac{2}{3}) + x = 5 + (-\frac{1}{2}x)$$

$$4 + (-\frac{2}{3}) + x + (-4) + \frac{2}{3} + \frac{1}{2}x = 5 + (-\frac{1}{2}x) + (-4) + \frac{2}{3} + \frac{1}{2}x$$

$$\frac{3}{2}x = \frac{5}{3}$$

$$\frac{2}{3}(\frac{3}{2}x) = \frac{2}{3}(\frac{5}{3})$$

$$\{\frac{10}{9}\}$$

(e) $\{\frac{27}{2}\}$

(f) all numbers less than 9

(g) all negative numbers

(h) ∅

5. (a) $15 + 8$
(b) $-25 + 4$
(c) $-9 + (-6)$
(d) $22 + 30$
(e) $-12 + 17$
(f) $8 + 5$
(g) $5 + (-10)$
(h) $7 + 8$ This deserves emphasis since it relates directly to the definition of subtraction.

6. If the bullet takes t seconds to reach the target, then the sound takes $2 - t$ seconds to return. Since the distances are equal, the open sentence is

$$\begin{aligned}
3300t &= 1100(2 - t) \\
3300t &= 1100(2) + 1100(-t) \\
3300t &= 2200 + (-1100t) \\
3300t + 1100t &= 2200 + (-1100t) + 1100t \\
4400t &= 2200 \\
\frac{1}{4400} \times 4400t &= \frac{1}{4400} \times 2200 \\
t &= \frac{2200}{4400} \\
t &= \frac{1}{2}
\end{aligned}$$

The time it took the bullet to reach the target was $\frac{1}{2}$ sec. The distance then is $\frac{1}{2} \times 3300$ or 1650 feet.

7. If r is the number of gallons of regular gas, then $500 - r$ is the number of gallons of ethyl.
The value of the regular is $(30r)$ cents.
The value of the ethyl is $35(500 - r)$ cents.
The value of the mixture is $32(500)$ cents.
Our sentence then is

The value of the regular plus the value of the ethyl is the value of the mixture.

or

$$\begin{aligned}
30r + 35(500 - r) &= 32(500) \\
30r + 17500 + (-35r) &= 16000 \\
(30)r + (-35)r + 17500 + (-17500) &= 16000 + (-17500)
\end{aligned}$$

$$30 + (-35)r = -1500$$
$$-5r = -1500$$
$$-\frac{1}{5}(-5)r = -\frac{1}{5}(-1500)$$
$$r = 300$$

The number of gallons of regular was 300.
There were 200 gallons of ethyl.

8. If t is the number of hours walked by the second man, then $t + 1$ is the number of hours walked by the first. The distance traveled by the first is $2(t + 1)$, by the second is $3t$. Since the distances walked were the same, our sentence is:

$$3t = 2(t + 1)$$
$$3t = 2t + 2$$
$$3t + (-2t) = 2t + (-2t) + 2$$
$$t = 2$$

The second man will have walked 2 hours when he catches up to the first.

10-2. Properties of Subtraction.

The title of this section might seem to be a misnomer, because we find that subtraction does not have many of the properties enjoyed by addition, such as the associative and commutative properties. The point is that we always change from indicated subtraction to addition and then apply the known properties of addition. Thus, multiplication appears to be distributive over subtraction:

$$a(b - c) = ab - ac$$

only because multiplication is distributive over addition:

$$a\left(b + (-c)\right) = ab + (-ac)$$

In this sense, subtraction can be thought of as having the properties of addition, but only because subtraction is defined as addition of the opposite.

Answers to Oral Exercises 10-2a; page 376:

1. (a) True
   (b) True
   (c) True
   (d) False
   (e) True
   (f) False

2. (a) -6
   (b) -14
   (c) -8
   (d) 3
   (e) 2
   (f) -3
   (g) $-14x$
   (h) $-a$
   (i) -42
   (j) $-10a$

Answers to Problem Set 10-2a; page 377:

1. (a) -58
   (b) $-9a$
   (c) $\frac{9}{2}x$
   (d) 0
   (e) $x - 5$
   (f) $-\frac{19}{3}y - \frac{3}{2}x$
   (g) $-6x^2 - 9x - 6xy$
   (h) $-6x^2 - 9x - 6xy$
   (i) $7m - 2x + 4$
   (j) $7m - 2x + 4$
   (k) $-3a + 5b$

2. The sentences in (a), (c), (e), and (i) are true for all values of the variables.

Answers to Oral Exercises 10-2b; page 381:

1. (a) $-b$ [also $(-1)b$]
   (b) c
   (c) $3c$
   (d) $-11x$
   (e) $-3x - 2$
   (f) $-2x + 5$
   (g) $-a - 2b + c$
   (h) $x - 2y$
   (i) $x - w$

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2. (a) 4
   (b) $x - y - z$
   (c) $a - x - 7$
   (d) $3x$
   (e) $-2x + 4$
   (f) $-a + 2$
   (g) $-3y + 5$

Answers to Problem Set 10-2b; pages 381-384:

1. (a) $-7 + 2x$
   (b) $-a - b$
   (c) $4 - 2c$
   (d) $10a$
   (e) $-1 + .01x$
   (f) $-\frac{3}{4}x + 2y + \frac{5}{4}x - 2y$ or $\frac{1}{2}x$
   (g) $-3x(2x + 3)$ or $-6x^2 - 9x$
   (h) $-7m(3m - 2)$ or $-21m^2 + 14m$

2. (a) $3 - x - 2$ or $-x + 1$ or $1 - 4$

   (b)
$$\begin{aligned} 2y + 5 - (5y - 3) &= 2y + 5 + \big(-(5y - 3)\big) \\ &= 2y + 5 + \big((-5y) + 3\big) \\ &= 2y + 5 + (-5y) + 3 \\ &= -3y + 8 \end{aligned}$$

   (c) $5a - 10$

   (d) $-5m + n$ (distributive property)

   (e)
$$\begin{aligned} -(5m - n) &= -5m + \big(-(-n)\big) && \text{(opposite of a sum)} \\ &= -5m + n && \text{(opposite of the opposite)} \end{aligned}$$

   (f) $7x + 3y - 4$

   (g) $-6x + 4b$

   (h) $-100x + x$ or $-99x$

   (i) $4y - 2$

   (j) $5t + 8$

   (k)
$$\begin{aligned} -x(x - y) &= (-x)x + (-x)(-y) \\ &= -(x)(x) + xy \\ &= -x^2 + xy \end{aligned}$$

(l) $(9a + 2b - 7) - (3a - 7b + 5) =$

$$9a + 2b - 7 + (-3a) + \big(-(-7b)\big) + (-5)$$

$$= 6a + 9b - 12$$

(m) $3x - x^2 - x(1 - x) = 3x - x^2 - x + x^2$

$$= 2x$$

(n) $(x - 1)(x + 1) - (x^2 - 1) = x^2 - 1 - (x^2 - 1)$

$$= 0$$

(o) $2m^2 - 6m(m - 1) - m = 2m^2 - 6m^2 + 6m - m$

$$= -4m^2 + 5m$$

(p) $(x + 2)(x + 1) - (x + 2)x = x^2 + 3x + 2 - x^2 - 2x$

$$= x + 2$$

Another way:

$(x + 2)(x + 1) - (x + 2)x = (x + 2)\big(x + 1 + (-x)\big)$

(distributive property)

$$= (x + 2)(1)$$

$$= x + 2$$

3. The sentences in (a), (b), (d), and (f) are true for all values of the variables.

4. (a) $\{-6\}$

(b) $\{-5\}$

(c) set of real numbers equal to or greater than 4

(d) set of real numbers less than 2

(e) set of real numbers equal to or greater than -3

(f) $\emptyset$

(g) $\{5\}$

(h) $\{2\}$

(i) $\emptyset$

(j) $\{\frac{1}{5}\}$

5. (a) $-2a^2 + 2b^2$

(b) $-3x + 5y$

(c) $9a + \frac{1}{4}b - 11$

(d) $k - 9k^2 - 29$

(e) $n^2 + 23n - 3$

6. (a) $n - 8$ if John is now n years old

   (b) $m = 6b$ if the boy is b years old and the man m years old

   (c) $5d = 36$ if d is the distance in miles

   (d) $\ell = 2w + 2$ if w is the number of feet in the width and ℓ is the number of feet in the length

   (e) $3(3y)$ or $9y$ feet

   (f) $(1.1)x$ if x is the number of pounds of candy

   (g) $30x + 35(x + 40)$ if x is the number of gallons of 30¢ gasoline

   (h) $100(2d)$

   (i) $15 + 2k$ if k is the number of dollars I have

7. $\left(2\left(\frac{7n + 12 - 4 + n}{8}\right) - 4\right)\frac{1}{2}$ is the form of the exercise.

   Simplifying,

$$\begin{aligned}\left(2\left(\frac{8n + 8}{8}\right) - 4\right)\frac{1}{2} &= \left(2(n + 1) - 4\right)\frac{1}{2}\\ &= (2n + 2 - 4)\frac{1}{2}\\ &= (2n - 2)\frac{1}{2}\\ &= n - 1\end{aligned}$$

   Thus $n - 1$ is the simplest general form.
   Starting with 2, $2 - 1 = 1$ is the final number.
   Starting with 11, $11 - 1 = 10$ is the final number.
   Starting with -3, $-3 - 1 = -4$ is the final number.

8. d is the number of dimes
   $d + 1$ is the number of quarters
   $2d + 1$ is the number of nickels

$$\begin{aligned}.10d + .05(2d + 1) + .25(d + 1) &= 1.65, \quad \text{or}\\ 10d + 5(2d + 1) + 25(d + 1) &= 165\\ 10d + 10d + 5 + 25d + 25 &= 165\\ 45d + 30 + (-30) &= 165 + (-30)\\ 45d &= 135\end{aligned}$$

$$\frac{1}{45} \cdot 45d = \frac{1}{45} \times 135$$
$$1 \cdot d = 3$$
$$d = 3$$

The number of quarters is 4.

9. Let n be the number of half-pint bottles.
Then $6n$ is the number of pint bottles.
39 quarts is the same as $2(39)$ pints.

$$\frac{1}{2}n + 6n = 2(39)$$
$$n + 12n = 4(39)$$
$$13n = 156$$
$$n = 12$$

There are 12 half-pint bottles.

10. $11a + 13b - 7c - (8a - 5b - 4c) = 11a + 13b - 7c - 8a + 5b + 4c$
$= 3a + 18b - 3c$

11. $-9s - 3u - (3s - 4t + 7u) = -9s - 3u - 3s + 4t - 7u$
$= -12s - 10u + 4t$

12. (i) If $a = b + c$
then $a + (-b) = b + c + (-b)$
and $a - b = c$

(ii) If $a - b = c$
then $a + (-b) = c$
and $a + (-b) + b = b + c$
and $a = b + c$

10-3. Subtraction in Terms of Distance.

The relation between the difference of two numbers and the distance between their points on the number line is introduced here to make good use again of the number line to help illustrate our ideas.

You are no doubt aware, however, of the fact that $(a - b)$ as a directed distance and $|a - b|$ as a distance are very helpful concepts in dealing with slope and distance in analytic geometry.

Answers to Oral Exercises 10-3; page 388:

1. (a) 8
(b) 8
(c) -8
(d) 8
(e) -4
(f) 4
(g) 6
(h) 6
(i) -7
(j) 7

2. (a) $5 - x$
(b) $|5 - x|$ (also $|x - 5|$)
(c) $x + 2$
(d) $|x + 2|$
(e) $-x + 1$
(f) $|-x + 1|$
(g) x
(h) $|x|$

Answers to Problem Set 10-3; pages 389-390:

1. (a) $1 - (-5) = 6$
(b) $-5 - 1 = -6$
(c) $|1 - (-5)| = 6$
(d) $|-5 - 1| = 6$
(e) $3 - 8 = -5$
(f) $8 - 3 = 5$
(g) 5
(h) 5

2. (a) $-3 -(-8) = 5$
(b) $7 - 4 = 3$
(c) $5 - 0 = 5$
(d) $6 - (-5) = 11$
(e) -8
(f) 1
(g) 4
(h) 8

3. (a) $x - 5$
(b) $5 < x$ is true
(c) $x < 5$ is true
(d) 9
(e) 1
(f) 1 or 9
(g) $\{1, 9\}$

4. (a) The information given can be translated:

$$|x - 5| < 4 \text{ and } x > 5.$$

Since $x > 5$, $|x - 5| = x - 5$.

x must be such that $x - 5 < 4$ and $x > 5$.

Hence x must be greater than 5 but less than 9.

(b) $|x - 5| < 4$ and $x < 5$

$x - 5 < 4$ tells us that x is between 1 and 9. But x shall be less than 5.

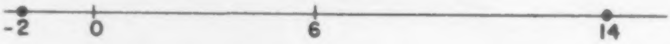
Hence x is between 1 and 5.

(c) x is between 1 and 9.

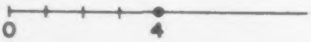
(d) The set of all numbers which are greater than 1 and less than 9

5. The sentence $|x - 4| = 1$ tells us that the distance between x and 4 on the number line is 1. The sentence is true when x is 3, also when x is 5.
Truth set: {3, 5}

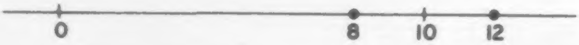
6. (a) {-2, 14}



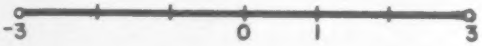
(b) {4}



(c) {8, 12}

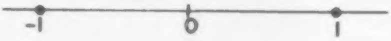


(d) all real numbers which are greater than -3 and less than 3



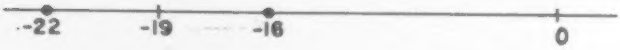
(e) the set of all real numbers

(f) {-1, 1}

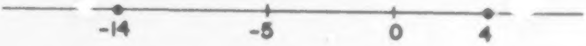


(g) $\emptyset$

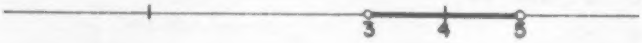
(h) {-22, -16}



(i) {-14, 4}



7. Truth set: the set of all numbers which are greater than 3 and less than 5



The graph consists of all points whose distance from 4 is less than 1.

8. The set of numbers which are either greater than 5 or less than 3.

The graph of $|x - 4| > 1$ would consist of all points whose distance from 4 is greater than 1.

9. The graph is the same as in problem 7. The truth sets are the same.

10.

$\|x\| = 3$	$x = -3$ or $x = 3$
$\|x\| < 3$	$x > -3$ and $x < 3$
$\|x\| \leq 3$	$x \geq -3$ and $x \leq 3$
$\|x\| > 3$	$x < -3$ or $x > 3$
$\|x\| \geq 3$	$x \leq -3$ or $x \geq 3$

10-4. Division.

In a manner analogous to the definition of subtraction in terms of addition, we define division by a non-zero number in terms of multiplication by its multiplicative inverse. The word "reciprocal" is introduced to mean the same thing as "multiplicative inverse". The symbol $\frac{1}{b}$ is introduced to represent the reciprocal of b, where b is a non-zero number.

At this point it might help to draw on the analogy between the reciprocal (or multiplicative inverse) and the opposite (or additive inverse).

Corresponding to each real number x there is a unique number y such that $x + y = 0$.

Corresponding to each non-zero number x there is a unique number y such that $xy = 1$.

This unique number y is called the opposite of the number x and is denoted by $-x$.	This unique number y is called the reciprocal of the number x and is denoted by $\frac{1}{x}$.
The opposite of the opposite of x is x: $-(-x) = x$.	The reciprocal of the reciprocal of x is x: $\frac{1}{\frac{1}{x}} = x$, if $x \neq 0$.
For real numbers a, b, c, $a - b = c$ if and only if $a = b + c$.	For real numbers a, b, c, with $b \neq 0$, $\frac{a}{b} = c$ if and only if $a = bc$.
The sum of the opposites is the opposite of the sum: $(-x) + (-y) = -(x + y)$	The product of the reciprocals is the reciprocal of the product: $(\frac{1}{x})(\frac{1}{y}) = \frac{1}{xy}$, if $x \neq 0$ and $y \neq 0$.

Again, like subtraction, the operation of division has no properties in its own right, but when written in terms of multiplication of the reciprocal it can be thought of as having all the properties of multiplication. Thus

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

can be thought of as a statement that division is distributive over addition, whereas in reality it is a statement that multiplication is distributive over addition:

$$\begin{aligned} \frac{a + b}{c} &= (a + b)\frac{1}{c} && \text{by definition of division} \\ &= a \cdot \frac{1}{c} + b \cdot \frac{1}{c} && \text{distributive property} \\ &= \frac{a}{c} + \frac{b}{c} && \text{by definition of division} \end{aligned}$$

page 391-395: 10-4

Answers to Oral Exercises 10-4a; pages 391-392:

1.	(a)	What number do we multiply by 4 to get 12?	3
	(b)	What number do we multiply by 4 to get -12?	-3
	(c)	What number do we multiply by -4 to get 12?	-3
	(d)	What number do we multiply by -4 to get -12?	3
	(e)	What number do we multiply by 4 to get $4a$?	a
	(f)	What number do we multiply by $3m$ to get $12m$?	4
	(g)	What number do we multiply by 9 to get $27x^2$?	$3x^2$
	(h)	What number do we multiply by -13 to get $26a$?	$-2a$
	(i)	What number do we multiply by $\frac{1}{2}$ to get 4?	8
	(j)	What number do we multiply by -2 to get $-6a$?	$3a$

2. (a) $\frac{b}{a}$

   (b) $\frac{a}{b}$

   (c) $\frac{b}{a}$

   (d) $\frac{a}{b}$

   (e) $\frac{a}{b}$

   For (a) and (c): "What number multiplied by a gives us the product b?"

   For the others: "What number multiplied by b gives us the product a?"

Answers to Problem Set 10-4a; pages 392-393:

1. -4
2. $2a$
3. -5
4. -7
5. $-5m$
6. -21
7. $-7a$
8. x
9. $5ax$
10. $a + b$
11. 1
12. $-3a$
13. x
14. $-4a$
15. 12
16. -1
17. 3
18. $-2a$
19. 0
20. -16

Answers to Oral Exercises 10-4b; page 395:

1. (a) $\frac{1}{5}$ (b) $-\frac{1}{7}$ (c) 2

(d) -6

(e) $\frac{5}{3}$

(f) -3

(g) $\frac{3}{7}$

(h) $-\frac{1}{12}$

(i) $\frac{1}{c}$, if $c \neq 0$

(j) d, d is assumed to be different from zero

2. (a) True
   (b) False
   (c) False
   (d) True
   (e) True
   (f) False

3. $\frac{3}{13}$ multiplied by $\frac{13}{3}$ yields 1 as product.

4. If n is a reciprocal of 0, then $0 \cdot n = 1$, because the product of a number and its reciprocal shall be 1. The sentence $0 \cdot n = 1$ has the empty set, $\emptyset$, as its truth set. There is no number which when multiplied by 0 yields 1.

Answers to Oral Exercises 10-4c; pages 399-400:

1. (a) True for all values of b except zero; $\frac{1}{7} \cdot 7 = 1$
   (b) Not true for any value of a
   (c) True for all values of b and c and all values of a except zero
   (d) True for all values of x and a

$$(-3)2 = -6\ ;\quad \left(-\frac{1}{3}\right)(-3)2 = \left(-\frac{1}{3}\right)(-6)$$

   (e) True

$$\frac{-24}{3} = -8\ ;\quad -24 = (-8)(3)$$

When we say "If $\frac{a}{b} = c$" we imply that $b \neq 0$.

   (f) True for all values of x except -3
   (g) True for all values of x

$$\frac{2(5)+3}{1} = 2(5) + 3$$

2. (a) $\frac{1}{15}$
   (b) $-\frac{1}{3}$
   (c) 3
   (d) $-\frac{3}{2}$

(e) $\frac{8}{3}$

(f) $\frac{1}{-.9}$, which is $-\frac{10}{9}$

(g) no multiplication required

3. (a) $\frac{5}{1} \cdot 1$ (d) $x(\frac{x}{1})$

(b) $6 \cdot \frac{4}{3}$ (e) $(-3y)(-\frac{3y}{2})$

(c) $(-7) \cdot (\frac{3}{2})$ (f) $(-\frac{2}{3})(\frac{5}{4})$

4. (a) $-\frac{1}{5}$ (d) -2

(b) -5 (e) a

(c) $\frac{1}{2}$

Answers to Problem Set 10-4c; pages 401-402:

1. (a) $\frac{5}{6} \cdot \frac{5}{2}$ (d) $\frac{a}{b} \cdot \frac{y}{x}$

(b) $\frac{3}{4} \cdot \frac{5}{1}$ (e) $\frac{a}{x} \cdot \frac{a}{x}$

(c) $a \cdot \frac{1}{b}$ (f) $\frac{3}{n - 1} \cdot \frac{a}{2}$

2. (a) $\frac{1}{13}$ (d) 3

(b) $-\frac{1}{5}$ (e) $-\frac{7}{5}$

(c) $\frac{4}{3}$ (f) $\frac{6}{15}$

3. (a) $\frac{42}{6}$ (d) $\frac{x}{y}$

(b) $\frac{5}{6}$ (e) $\frac{bc}{\frac{3}{2}}$ (also $\frac{2bc}{3}$)

(c) $\frac{c}{2}$ (f) $\frac{5}{\frac{1}{6}}$ $\left(\text{(also } \frac{6}{\frac{1}{5}}\text{)}\right)$

4. (a) $\{\frac{41}{6}\}$ (b) $\{\frac{71}{2}\}$

(c) $-5m = -16$

$m = \frac{-16}{-5}$

$= \frac{16}{5}$

Truth set: $\{\frac{16}{5}\}$

(d) $\{-\frac{7}{4}\}$

(e) $\{-25\}$

(f) $\{-48\}$

5. (a) $6y = 41$

$\frac{1}{6}(6y) = \frac{1}{6}(41)$

$\frac{1}{6}(6)y = \frac{1}{6}(41)$

$y = \frac{41}{6}$

$\frac{1}{a} \cdot b = \frac{b}{a}$

(b) $2x = 71$

$\frac{1}{2}(2x) = \frac{1}{2}(71)$

$x = \frac{71}{2}$

6. $m + (m + 2) \leq 83$

$2m \leq 81$

$m \leq \frac{81}{2}$

The first number may be any odd integer from 1 to 39 inclusive. The second is two more than the first.

7. Let x be the width in inches.
Then 7x is the length in inches.

$2x + 14x = 144$

$x = 9$

$7x = 63$

The width is 9 inches.

8. Let x be Dick's age in years.
Then 3x is John's age in years.

$(x - 3) + (3x - 3) = 22$

$x = 7$

$3x = 21$

Dick is 7 and John is 21

page 402: 10-4

9. $n + (n + 2) = 46$

$n = 22$

$n + 2 = 24$

The numbers are 22 and 24

10. $\frac{1}{2}x = \frac{1}{5}x + 3$

$x = 9$

The number is 9.

11. Let the speed of the wind be x miles per hour.
The speed of the plane is 200 - x miles per hour.

$(3\frac{1}{2})(200 - x) = 630$

$x = 20$

The average speed of the wind is 20 mph.

12. (i) Prove: If $a = bc$ and $b \neq 0$, then $\frac{a}{b} = c$.

Proof: If $b \neq 0$ then $\frac{1}{b}$ is a real number.

Then if $a = bc$,

$a \cdot \frac{1}{b} = (bc) \cdot \frac{1}{b}$ multiplicative property of equality

$\frac{a}{b} = (b \cdot \frac{1}{b})\, c$ associative and commutative properties of multiplication, and definition of division

$\frac{a}{b} = 1 \cdot c$

$\frac{a}{b} = c.$

(ii) Prove: If $\frac{a}{b} = c$ and $b \neq 0$, then $a = bc$.

Proof: If $\frac{a}{b} = c$, then

$a \cdot \frac{1}{b} = c$ definition of division

$b(a \cdot \frac{1}{b}) = bc$ multiplicative property of equality

$(b \cdot \frac{1}{b})a = bc$ associative and commutative properties of multiplication

$(1)a = bc$

$a = bc.$

Answers to Oral Exercises 10-4d; pages 404-405:

1. (a) $\frac{2}{-13}, -\frac{2}{13}$

   (b) $\frac{-3}{11}, -\frac{3}{11}$

   (c) $\frac{-4}{3}, \frac{4}{-3}$

   (d) $\frac{5}{-7a}, -\frac{5}{7a}$

   (e) $\frac{-6}{7a}, -\frac{6}{7a}$

   (f) $\frac{3m}{-2n}, -\frac{3m}{2n}$

   (also $\frac{3(-m)}{2n}$, $\frac{-3m}{2n}$, etc.)

2. (a) $\frac{8}{9}$

   (b) $\frac{6}{11a}$

   (c) $\frac{2a}{3b}$

   (d) $\frac{3x}{7y}$

   (e) $\frac{5a}{4}$

   (f) $\frac{8m}{7n}$

3. (a) $\frac{5}{3} + \frac{-2}{3} = \frac{3}{3}$

   (b) $$\begin{aligned}\frac{3}{7} - \frac{2}{-7} &= \frac{3}{7} - \left(-\frac{2}{7}\right)\\ &= \frac{3}{7} + \frac{2}{7}\\ &= \frac{5}{7}\end{aligned}$$

   (c) $$\begin{aligned}\frac{3}{7} + \frac{2}{-7} &= \frac{3}{7} - \frac{2}{7}\\ &= \frac{1}{7}\end{aligned}$$

   (d) $$\begin{aligned}\frac{3a}{b} - \frac{a}{-b} &= \frac{3a}{b} + \frac{a}{b}\\ &= 3a \cdot \frac{1}{b} + a \cdot \frac{1}{b}\\ &= (3a + a)\frac{1}{b}\\ &= 4a \cdot \frac{1}{b}\\ &= \frac{4a}{b}\end{aligned}$$

   (e) $\frac{4}{-7} + \frac{-4}{7} = \frac{-4}{7} - \frac{4}{7} = -\frac{8}{7}$

   (f) $-\frac{3}{2x} + \frac{5}{2x} = \frac{2}{2x} = \frac{1}{x}$

Answers to Problem Set 10-4d; pages 405-406:

1. (a) $\frac{3}{-5}, -\frac{3}{5}$

   (b) $\frac{-1}{7}, -\frac{1}{7}$

   (c) $\frac{-2}{3}, \frac{2}{-3}$

   (d) $\frac{-5}{6}, \frac{5}{-6}$

   (e) $\frac{-7}{3}, -\frac{7}{3}$

   (f) $\frac{7}{-4}, -\frac{7}{4}$

   (g) $\frac{3a}{-4b}, -\frac{3a}{4b}$

   (h) $\frac{-2m}{3n}, \frac{2m}{-3n}$

   (i) $\frac{-5}{6a}, -\frac{5}{6a}$

2. (a) $-\frac{3a}{4b}$
   (b) $-\frac{2m}{3n}$
   (c) $-\frac{5}{6a}$
   (d) $-\frac{16}{3x}$
   (e) $-\frac{13a}{2k}$
   (f) $-\frac{ab}{c}$

3. (a) $\frac{1}{2}$
   (b) $\frac{3}{5}$
   (c) $-\frac{10}{3}$
   (d) 0
   (e) $-\frac{8}{5}$
   (f) 1

4. (a) $\frac{5}{8}+\left(\frac{-1}{4}\right) = \frac{5}{8}+\left(\frac{-2}{8}\right)$
   $= \frac{3}{8}$
   (b) $\frac{1}{12}+\left(\frac{-8}{12}\right) = \frac{-7}{12}$
   $= -\frac{7}{12}$
   (c) $-\frac{2}{a}$
   (d) $\frac{2}{3b}+\left(\frac{-a}{3b}\right) = \frac{2-a}{3b}$
   (e) $-\frac{2a}{b}$
   (f) $\frac{-5}{3m}+\frac{-2a}{3m} = \frac{-5-2a}{3m}$
   or $-\frac{5+2a}{3m}$

5. (a) $\frac{3}{4}$
   (b) $\frac{8}{13}$
   (c) $\frac{2}{3}$
   (d) $\frac{5}{a}$
   (e) $\frac{2a}{3b}$
   (f) $\frac{3}{5x}$

6. (a) 1
   (b) $\frac{8}{7}$
   (c) 4
   (d) $\frac{4}{a}$
   (e) $\frac{8b}{5c}$
   (f) 0

7. (a) $\frac{1}{a-b}+\frac{-2}{a-b} = -\frac{1}{a-b}$
   (b) $\frac{5}{x-y}$
   (c) $\frac{3a}{a-b}$, or $-\frac{3a}{b-a}$
   (d) $\frac{4a}{m+n}$
   (e) $\frac{8a}{m+n}$
   (f) $\frac{a+b}{a-b}$

10-5. Common Names.

In this and the following section we are interested in three commonly accepted conventions about the simplest numeral for a number.

(1) There should be no indicated operations remaining which can be performed.

(2) If there is an indicated division, the numbers whose division is indicated should have no common factor.

(3) We prefer $-\frac{a}{b}$ to $\frac{-a}{b}$ or $\frac{a}{-b}$.

Thus, to illustrate the first convention we would say that "$\frac{20}{4}$" is not as simple as "5"; "$\frac{2+3}{6}$" is not as simple as "$\frac{5}{6}$"; "$\frac{3 \cdot 5}{7}$" is not as simple as "$\frac{15}{7}$"; but "$\frac{x+3}{y}$" cannot be simplified. Similarly, for the second convention "$\frac{14}{21}$" is not as simple as "$\frac{2}{3}$" and "$\frac{2x^2+4}{ax^2+2a}$" is not as simple as "$\frac{2}{a}$". Simplifications of this kind depend on the theorem which states that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, the fact that $\frac{a}{a} = 1$ for $a \neq 0$, and the property of 1.

The student has for years been "multiplying fractions" according to the theorem proved here. It is not a new result to him, but it is now a consequence of and is directly tied to our definition of division and the properties of multiplication. In the past he knew how to divide; now he learns why he divides in this manner.

In the process of proving the theorem, it must be established that $(\frac{1}{b})(\frac{1}{d}) = \frac{1}{bd}$, that is, that the product of the reciprocals of two non-zero numbers is the reciprocal of the product of the numbers. To do this, use the commutative and associative properties of multiplication:

$$\begin{aligned} bd(\tfrac{1}{b})(\tfrac{1}{d}) &= (b \cdot \tfrac{1}{b})(d \cdot \tfrac{1}{d}) \\ &= (1)(1) \\ &= 1. \end{aligned}$$

Hence, $(\frac{1}{b})(\frac{1}{d})$ is the reciprocal of bd; i.e.,

$$(\tfrac{1}{b})(\tfrac{1}{d}) = \tfrac{1}{bd}.$$

Notice that we have become relaxed in our use of the words "numerator" and "denominator". Although these words refer to numerals, we shall begin to use them interchangeably for numerals and numbers, whenever the context is clear.

Answers to Oral Exercises 10-5; page 410:

1. (a) $\frac{1}{2}(\frac{2}{2}) = \frac{1}{2}$

(b) $\frac{7}{8}(\frac{2}{2}) = \frac{7}{8}$

(c) $\frac{x}{y}(\frac{3}{3}) = \frac{x}{y}$, $y \neq 0$

(d) $\frac{-1}{x}(\frac{x}{x}) = -\frac{1}{x}$, $x \neq 0$

(e) $\frac{x+2}{x-1}(\frac{3}{3}) = \frac{x+2}{x-1}$, $x \neq 1$

(f) $\frac{2}{6}(\frac{x+2}{x+2}) = \frac{1}{3}$, $x \neq -2$

(g) $\frac{2}{3}(\frac{x-2}{x-2}) = \frac{2}{3}$, $x \neq 2$

(h) This is in simplest form.

(i) This is in simplest form.

(j) $(x+2)(\frac{x+3}{x+3}) = x+2$, $x \neq -3$

Answers to Problem Set 10-5; pages 411-412:

1. (a) $\frac{2}{7a}$, $a \neq 0$

(b) $\frac{a}{b}$, $b \neq 0$

(c) $\frac{-3y}{m}$, $x \neq 0$ and $m \neq 0$ ($xm \neq 0$ says the same thing as $x \neq 0$ and $m \neq 0$)

(d) $\frac{3}{x}$, $x \neq 0$

(e) [illegible] , $x \neq -2$

(f) is in simplest form, $x \neq -\frac{1}{2}$

(g) $x - 3$, $x \neq -2$ and $x \neq 3$

(h) $(x-2)(x+4)$, $x \neq 0$

(i) $\frac{b+1}{b}$, $b \neq 0$ and $x \neq 0$

(j) $\frac{3}{5}$, $x \neq -1$

(k) $2 - a$, $a \neq 0$

(l) $x + 1$, $y \neq 0$

(m) is in simplest form, $y \neq \frac{3}{2}$

(n) $\frac{1}{3}$, $x \neq 1$

(o) -1, $b - a \neq 0$

(p) -1, $x \neq 2$

page 411: 10-5

2. (a) $\frac{3x}{3x}(3x) = 6$ (Zero is excluded from the domain)

$$3x = 6$$

$\{2\}$

(b) The value 1 is excluded from the domain since the left side of the sentence is meaningless for $x = 1$.

If $x \neq 1$, $\frac{x - 1}{x - 1} = 1$.

Thus the sentence $\frac{3x(x - 1)}{x - 1} = 3$

becomes $3x = 3$,

which is equivalent to $x = 1$.

The truth set is empty since 1 is not in the domain.

(c) Excluding the value -1 from the domain of x, the given sentence is equivalent to

$$3x = 3,$$

which is equivalent to $x = 1$.

The truth set is $\{1\}$.

(d) $\{0\}$ Exclude the value -1 from the domain of x.

*(e) Exclude the value 1 from the domain of x.

$$\frac{6x - 6}{4x - 4} < 5$$

$$\frac{6(x - 1)}{4(x - 1)} < 5$$

$$\frac{6}{4} < 5$$

This means that the truth set consists of all real numbers except 1.

*(f) Exclude 0 and 7 from the domain of x.

$$\frac{-x + 7}{3x(7 - x)} = 3x$$

$$\frac{1}{3x} = 3x$$

Truth set $\{\frac{1}{3}, -\frac{1}{3}\}$. 1 and -1 are the only numbers which are their own reciprocals.

pages 411-412: 10-5 and 10-6

*(g) $\frac{7x - 2 + 3x - 4}{-8} = -\frac{1}{2}$ No numbers need be excluded from the domain of x.

$$\frac{10x - 6}{-8} = -\frac{1}{2}$$

$$10x - 6 = 4$$

$$\{1\}$$

3. Let t be the required number of years.
After t years Brown's salary will be $3600 + 300t$.
After t years Jones' salary will be $4500 + 200t$.

$$3600 + 300t = 4500 + 200t$$

$$\{9\}$$

After 9 years their salaries will be the same.

4. Let x be Bill's age in years now.
Then $2x$ is Bob's age in years now.

$$(x + 3) + (2x + 3) = 30$$

$$x = 8$$

$$2x = 16$$

Bill is 8 years old. Bob is 16 years old.

10-6. Fractions.

The main point of this section is to develop skill in simplifying an expression to one in which there is at most one indicated division. This essentially means that we are learning to multiply, divide, and add fractions.

Page 412. We are again relaxing our rigor in the use of words. We shall allow ourselves to use "fraction" for either the symbol or the number, even though correctly speaking it means the symbol. Thus in the preceding paragraph a precise statement would have said "to multiply, divide, and add numbers which are represented by fractions."

Now that we have begun to relax our precision of language, we shall hereafter, without further comment, feel free to use convenience of language even when it violates precision of language about numbers and numerals, as long as we are sure that the precise meaning will be understood.

pages 412-415: 10-6

We define the word "ratio" (Problem Set 10-6b) as part of the language in certain applications. We also call an equation such as $\frac{x}{1197} = \frac{z}{19}$, which equates two ratios, a "proportion". It seems undesirable at present to digress into a lengthy treatment of ratio and proportion since it is just a matter of special names for familiar concepts.

__Answers to Oral Exercises 10-6a__; pages 414-415:

1. $-\frac{1}{7}$
2. $\frac{1}{3}$
3. $-\frac{3}{5}$
4. $-\frac{4x}{3}$
5. $\frac{7x}{6}$
6. $\frac{x^2 + x}{3}$
7. $x - 1, x \neq 0$
8. $x - 3, \ x \neq 1$
9. $-\frac{12x}{x - 1}$ or $\frac{12x}{1 - x}$, $x \neq 1$
10. $\frac{1}{3}$, $x \neq -1$
11. $-1, \ x \neq 1$
12. $-1, \ a \neq -b$
13. $6a, \ a \neq 2, a \neq 0$
14. is in simplest form, $y \neq -1$

__Answers to Problem Set 10-6a__; pages 415-417:

1. (a) $\frac{3}{8} \cdot \frac{7}{2} = \frac{3 \cdot 7}{8 \cdot 2} = \frac{21}{16}$

   (b) $\frac{4}{7} \cdot \frac{21}{10} = \frac{4 \cdot 21}{7 \cdot 10}$
   $= \frac{2 \cdot 2 \cdot 7 \cdot 3}{7 \cdot 2 \cdot 5}$
   $= \frac{7}{7} \cdot \frac{2}{2} \cdot \frac{2 \cdot 3}{5}$
   $= \frac{6}{5}$

   (c) $\frac{10}{9}$

   (d) $\frac{1}{n^2}$, $n \neq 0$

   (e) $1, \ n \neq 0$

   (f) $\frac{1}{nx}$, $nx \neq 0$

   (g) $\frac{1}{2n}$, $n \neq 0$

   (h) $\frac{x^2}{12}$

   (i) 5

   (j) $-\frac{29}{6}$

   (k) $\frac{4a^3}{3}$

   (l) $\frac{3}{x}$, $x \neq 0$
   already in simplest form

pages 415-417: 10-6

2. (a) $\frac{1}{5}$

(b) $\frac{1}{a}$, $a \neq 0$, $b \neq 0$, and $c \neq 0$

(c) $\frac{42}{55}$, $xy \neq 0$

(d) $\frac{1}{9}$, $bc \neq 0$

(e) 21, $ab \neq 0$

(f) $\frac{5}{7}$, $rs \neq 0$

(g) $\frac{x + y}{2}$, $x \neq y$

3. (a) $\frac{2(u + v)}{5(u + v)} = \frac{2}{5}$, $u \neq -v$

(b) $\frac{c(m + n)}{3(m + n)} = \frac{c}{3}$, $m \neq -n$

(c) $\frac{a(x - y)}{b(x - y)} = \frac{a}{b}$, $x \neq y$

(d) $\frac{2(x - 2)}{-3(x - 2)} = -\frac{2}{3}$, $x \neq 2$

(e) $\frac{3(3x - 2)}{5(3x - 2)} = \frac{3}{5}$, $x \neq \frac{2}{3}$

(f) $\frac{m(7m + 2)}{3(7m + 2)} = \frac{m}{3}$, $m \neq -\frac{2}{7}$

4. (a) $3x + \frac{1}{7} = \frac{8}{7}$

$\{\frac{1}{3}\}$

(b) $\{\frac{1}{4}\}$

(c) $\{\frac{1}{3}\}$

(d) $x > -\frac{1}{2}$

Truth set: all real numbers greater than $-\frac{1}{2}$

(e) $\{-\frac{1}{3}, \frac{1}{3}\}$

(f) $\{3\}$

5. If the freight train averages p miles per hour, then the passenger train averages $p + 20$ miles per hour. The distance traveled by the passenger train is $5(p + 20)$ miles; by the freight train is $5p$ miles.

$$5(p + 20) = 5p + 100$$
$$5p + 100 = 5p + 100$$

The sentence is true for every value of p. The freight train could have traveled at any speed whatsoever.

6. If $\$c$ was the price of the chair before the sale, then the discount was $(.2)c$.
The sale price was the original price less the discount, giving the sentence

$$c - .2c = 30$$
$$.8c = 30$$
$$c = 30(\tfrac{10}{8}) = \tfrac{300}{8}$$
$$= 37.50$$

The original cost of the chair was \$37.50.

7. $\frac{1}{2}$ a number n is the same as $\frac{1}{6}n + 3$.

$$\frac{1}{2}n = \frac{1}{6}n + 3$$
$$3n = n + 18$$
$$2n = 18$$
$$n = 9$$

The number is 9.

Page 418. The reasons for the steps are:

1. Definition of subtraction
2. $-\frac{a}{b} = \frac{-a}{b}$
3. Multiplication property of one
4. $\frac{a}{a} = 1$, if $a \neq 0$
5. Theorem on multiplication of fractions
6. Definition of division: $\frac{a}{b} = a \cdot \frac{1}{b}$
7. Distributive property
8. Definition of division, and $\frac{-a}{b} = -\frac{a}{b}$.

The number of steps needed to do such a simplification will vary from student to student. After he understands reasons for every step, he will soon be able to write

$$\frac{5x}{9} - \frac{2x}{3} = \frac{5x}{9} - \frac{6x}{9}$$
$$= (5x - 6x)\frac{1}{9}$$
$$= -\frac{x}{9}\ .$$

Answers to Oral Exercises 10-6b; pages 420-421:

1. (a) $\frac{2}{2}$; $6x$
   (b) $\frac{8}{8}$; $40x$
   (c) $\frac{3}{3}$; 9
   (d) $\frac{-a}{-a}$; $-5a^2$
   (e) $\frac{3}{3}$; $21a$
   (f) $\frac{a + b}{a + b}$; $5(a + b)$

(g) $\frac{m + n}{m + n}$; $6a(m + n)$

(h) $\frac{b(x - y)}{b(x - y)}$; $b^2(x - y)^2$

(i) $\frac{-1}{-1}$; -4

(j) $\frac{-1}{-1}$; $x - 2$

2. (a) x

(b) $\frac{a}{3}$

(c) $\frac{4m + n}{5}$

(d) $\frac{12x + 1}{4}$

(e) $\frac{-4y - 10}{11}$

(f) $\frac{13}{6}$

(g) $\frac{10 + y}{8x}$

(h) $\frac{1 - 3y}{x + y}$

(i) $\frac{3x}{4}$

(j) $\frac{10}{3x}$

Answers to Problem Set 10-6b; pages 421-424:

1. (a) $-\frac{1}{9}$

(b) $\frac{9}{a}$

(c) $\frac{13}{2a}$

(d) $\frac{4a + 5}{a^2}$

(e) $\frac{19a}{35}$

(f) $\frac{20 - a}{35}$

(g) $\frac{3x - 6}{5}$

2. (a) 1

(b) $\frac{3x + 2y}{4}$

(c) 0

(d) $\frac{6x + y}{14}$

(e) $-\frac{5a}{36}$

(f) $\frac{x^2 + 1}{x}$

(g) $\frac{ab + b^2 + 1}{a + b}$

(h) $\frac{7 + 5(a + b)}{a + b} = \frac{7 + 5a + 5b}{a + b}$

(i) $\frac{15}{8x}$

(j) $\frac{-1}{x + 1}$

3. (a) $15(\frac{x}{5} + 3) = 15(\frac{2}{3})$

$3x + 45 = 10$

$\{-\frac{35}{3}\}$

(b) $\{\frac{3}{2}\}$

(c) $\{\frac{15}{2}\}$

(d) $\{0\}$

(e) all numbers greater than 1

pages 422-423: 10-6

(f) $\{-11\}$

(g) $\{\frac{2}{7}\}$

(h) $\{-10\}$

(i) $\emptyset$

(j) $\{-\frac{15}{2}\}$

4. Let s be the number of 4¢ stamps Mary bought. If she was charged the correct amount, then s must be a non-negative integer and

$$15(.03) + .04s = 1.80$$

If there is a non-negative integer s for which the above sentence is true, then each of the following sentences are true for this value of s.

$$\begin{aligned} .45 + .04s &= 1.80 \\ .04s &= 1.35 \\ 4s &= 135 \\ s &= \frac{135}{4} = 33\tfrac{3}{4} \end{aligned}$$

Since $33\frac{3}{4}$ is not an integer, she was charged the wrong amount.

5. He had 12 pennies, 16 dimes, 22 nickels. He has $2.82.

6. John has $25.

7. If one number is y, the other number is 240 - y.

$$\begin{aligned} y &= \tfrac{3}{5}(240 - y) \\ 5y &= 720 - 3y \\ y &= 90 \end{aligned}$$

The numbers are 90 and 150.

It should be pointed out that another open sentence for this problem is $240 - y = \frac{3}{5}y$. Here you get $y = 150$ and $240 - y = 90$. Again you get 90 and 150 as the two numbers.

8. $\frac{4 + x}{7} = \frac{27}{21}$

$12 + 3x = 27$

$x = 5$

The numerator was increased by 5.

9. 312

10. If his father is x years old, Joe is $\frac{1}{3}x$ years old.

$$\frac{1}{3}x + 12 = \frac{1}{2}(x + 12)$$
$$x = 36$$

The father's age is 36; the son's age is 12.

11. Let x be the smaller of the two numbers. Then the larger number is $7 - x$.

$$(7 - x) - x = 3$$
$$x = 2$$

The numbers: 2, 5

The sum of the reciprocals: $\frac{1}{2} + \frac{1}{5} = \frac{7}{10}$

The difference of the reciprocals: $\frac{1}{2} - \frac{1}{5} = \frac{3}{10}$

12. (a) If there were g girls,
there were $(2600 - g)$ boys.

$$\frac{2600 - g}{g} = \frac{7}{6}$$
$$6(2600 - g) = 7g$$
$$15600 - 6g = 7g$$
$$15600 = 13g$$
$$1200 = g$$

If $g = 1200$, $2600 - g = 1400$,

and $\frac{1400}{1200} = \frac{7}{6}$

Hence, there were 1200 girls in the school.

(b) $\frac{1}{20} \cdot 800$ or 40 radios were defective.

800 - 40 or 760 radios were not defective.

$\frac{40}{760} = \frac{1}{19}$ is the ratio of defective to non-defective radios.

The alert student will notice that the number 800 is unnecessary information. If we suppose that there were r radios in the shipment, then

$\frac{1}{20}r$ radios were defective, and

$\frac{19}{20}r$ radios were not defective.

$$\frac{\frac{1}{20}r}{\frac{19}{20}r} = \frac{1}{19}$$

Therefore the required ratio is $\frac{1}{19}$.

(c) Let f be the number of faculty members.

$$\frac{f}{1197} = \frac{2}{19}$$

$$f = \frac{2}{19} \cdot 1197$$

$$f = 126$$

Hence, there are 126 faculty members.

(d) Since $\frac{5x}{9x} = \frac{5}{9} \cdot \frac{x}{x} = \frac{5}{9} \cdot 1 = \frac{5}{9}$ (when $x \neq 0$), $5x$ and $9x$ are in the ratio of 5 to 9. More precisely, if $5x$ is the first of the numbers that are in the ratio $\frac{5}{9}$, and y is the other number, then

$$\frac{5x}{y} = \frac{5}{9}.$$

Then $$9y\left(\frac{5x}{y}\right) = 9y\left(\frac{5}{9}\right),$$

$$9(5x) = 5y,$$

and $$9x = y.$$

Hence, the numbers can be represented by $5x$ and $9x$, $x \neq 0$.

If $x = 7$, the numbers are 35 and 63.

If $x = 100$, the numbers are 500 and 900.

(e) If $\frac{a}{b} = \frac{c}{d}$, then

$(\frac{a}{b})bd = (\frac{c}{d})bd$	Multiplication property of equality
$(a \cdot \frac{1}{b})bd = (c \cdot \frac{1}{d})bd$	Definition of division
$(ad)(b \cdot \frac{1}{b}) = (bc)(d \cdot \frac{1}{d})$	Associative and commutative properties
$(ad) \cdot 1 = (bc) \cdot 1$	Definition of multiplicative inverse
$ad = bc$	Multiplication property of one

(f) If $ad = bc$ and $b \neq 0$ and $d \neq 0$, then

$(ad)(\frac{1}{b} \cdot \frac{1}{d}) = (bc)(\frac{1}{b} \cdot \frac{1}{d})$	Multiplication property of equality
$(d \cdot \frac{1}{d})(a \cdot \frac{1}{b}) = (b \cdot \frac{1}{b})(c \cdot \frac{1}{d})$	Associative and commutative properties of multiplication
$1 \cdot (a \cdot \frac{1}{b}) = 1 \cdot (c \cdot \frac{1}{d})$	Definition of multiplicative inverse
$a \cdot \frac{1}{b} = c \cdot \frac{1}{d}$	Multiplication property of one
$\frac{a}{b} = \frac{c}{d}$	Definition of division

(g) $\frac{6}{15} = \frac{2 \cdot 3}{5 \cdot 3} = \frac{2}{5} \cdot 1 = \frac{2}{5}$.

(h) $\frac{q}{y} = \frac{p}{x}$; $\frac{x}{p} = \frac{y}{q}$; $\frac{q}{p} = \frac{y}{x}$; $\frac{y}{x} = \frac{q}{p}$;

$\frac{p}{x} = \frac{q}{y}$; $\frac{y}{q} = \frac{x}{p}$; $\frac{p}{q} = \frac{x}{y}$

Answers to Oral Exercises 10-6c; page 427:

1. (a) $\frac{4}{4}$; also $\dfrac{\frac{4}{3}}{\frac{4}{3}}$

   (b) $\frac{bd}{bd}$; also $\dfrac{\frac{d}{c}}{\frac{d}{c}}$

   (c) $\frac{6}{6}$; also $\dfrac{\frac{3}{y}}{\frac{3}{y}}$

   (d) $\frac{15x}{15x}$; also $\dfrac{\frac{5x}{2}}{\frac{5x}{2}}$

   (e) $\frac{12xy}{12xy}$; also $\dfrac{\frac{4x}{3y}}{\frac{4x}{3y}}$

   (f) $\frac{28b}{28b}$; also $\dfrac{\frac{7b}{-2a}}{\frac{7b}{-2a}}$

   (g) $\dfrac{-\frac{bm}{7}}{-\frac{bm}{7}}$; also $\frac{3bm}{3bm}$

   (h) $\dfrac{\frac{6}{x-1}}{\frac{6}{x-1}}$; also $\frac{6}{6}$

   (i) $\frac{3x}{3x}$; also $\dfrac{\frac{x}{x-2}}{\frac{x}{x-2}}$

   (j) $\dfrac{\frac{x-1}{3x}}{\frac{x-1}{3x}}$; also

   $\frac{(x-3)(x-1)}{(x-3)(x-1)}$

Answers to Problem Set 10-6c; pages 427-431:

1. (a) $$\dfrac{\frac{2}{3}}{\frac{5}{6}} \cdot \dfrac{\frac{6}{5}}{\frac{6}{5}} = \dfrac{\frac{2}{3}\cdot\frac{6}{5}}{1}$$
   $$= \frac{12}{15}$$
   $$= \frac{4}{5}$$

   (b) 16

   (c) $\frac{9}{7}$

   (d) $\frac{4}{3}$

   (e) $\frac{10}{3}$

   (f) $\frac{9}{8}$

2. (a) $$\dfrac{\frac{2x}{3}}{\frac{5x}{2}} \cdot \frac{2}{2} = \dfrac{\frac{2x}{3}\cdot 2}{\frac{5x}{2}\cdot 2} = \dfrac{\frac{4x}{3}}{5x} = \frac{4x}{3}\cdot\frac{1}{5x} = \frac{4}{15}$$

   (b) $\frac{6}{5}$

   (c) 2

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(d) $\frac{b}{a}$

(e) 4

(f) $\dfrac{\frac{1}{x - 1}}{\frac{1}{x - 2}} \cdot \dfrac{x - 2}{x - 2} = \dfrac{\frac{x - 2}{x - 1}}{\frac{x - 2}{x - 2}} = \dfrac{x - 2}{x - 1}$

3. (a) $\dfrac{\frac{1}{2}x + \frac{1}{4}}{\frac{2}{3}} \cdot \dfrac{12}{12} = \dfrac{12(\frac{1}{2}x + \frac{1}{4})}{12(\frac{2}{3})} = \dfrac{6x + 3}{8}$

(b) $\dfrac{3(2x - 1)}{2(x + 1)} = \dfrac{6x - 3}{2x + 2}$

(c) 8

(d) $\frac{ab}{3}$

(e) $\frac{m}{6n}$

(f) $\dfrac{3(x - 1)}{2(x + 1)} = \dfrac{3x - 3}{2x + 2}$

4. (a) $\frac{9}{4}$

(b) $- \frac{11a}{12}$

(c) $-6a^2b^2x$

(d) $-a + 2b - 2c$

(e) $\dfrac{1}{2(x + 1)}$

(f) $\frac{7a}{3b}$

5. (a) $x^2 - 4$

(b) $3y - 6$, or $3(y - 2)$

(c) $- \frac{1}{8}x - \frac{3}{5}y + 12$

(d) $\dfrac{20x - 3}{7}$

(e) $\dfrac{20a}{3 + 2a}$

(f)
$$\dfrac{\frac{15y}{12} - \frac{8y}{12}}{\frac{4}{15} + \frac{3y}{15}} = \dfrac{\frac{7y}{12}}{\frac{4 + 3y}{15}}$$

$$= \dfrac{\frac{7y}{12} \cdot \frac{15}{4 + 3y}}{\frac{4 + 3y}{15} \cdot \frac{15}{4 + 3y}}$$

$$= \dfrac{35y}{16 + 12y}$$

6. (a) $\{4, -8\}$ (f) $\{\frac{4}{3}\}$

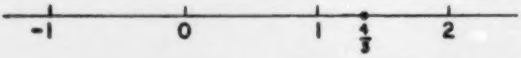
(b) $\emptyset$ (g) the set of all real numbers

(c) $\{\frac{5}{4}\}$ (h) $\{-\frac{98}{15}\}$

(d) $\{\frac{8}{5}\}$ (i) all real numbers except -4

(e) $\{\frac{79}{2}\}$ (j) $\{0, 3, -2\}$

7. (f)



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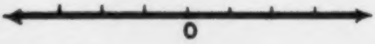

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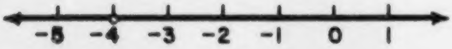
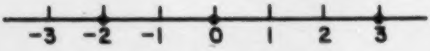


(j)



8. If the successive positive integers are n, $n + 1$, and $n + 2$, then the sentence is

$$\begin{aligned} n + (n + 1) + (n + 2) &= 1080 \\ 3n + 3 &= 1080 \\ 3n &= 1077 \\ n &= 359 \end{aligned}$$

The integers are 359, 360, and 361.

9. Let the successive positive integers be n and $n + 1$. Then

$$\begin{aligned} n + (n + 1) &< 25 \\ 2n + 1 &< 25 \\ 2n &< 24 \\ n &< 12 \end{aligned}$$

The numbers could be any of the pairs (11, 12), (10, 11), . . ., (1, 2).

10. If the consecutive even integers are n and $n + 2$, then

$$\begin{aligned} n + (n + 2) &= 46 \\ 2n + 2 &= 46 \\ 2n &= 44 \\ n &= 22 \end{aligned}$$

The numbers are 22 and 24.

11. If the whole number and its successor are n and $n + 1$, then

$$\begin{aligned} n + (n + 1) &= 45 \\ 2n + 1 &= 45 \\ 2n &= 44 \\ n &= 22 \end{aligned}$$

The numbers are 22 and 23.

12. If the two consecutive odd numbers are n and $n + 2$, then

$$\begin{aligned} n + (n + 2) &= 75 \\ 2n + 2 &= 75 \\ 2n &= 73 \\ n &= \frac{73}{2} \end{aligned}$$

But n must be an integer, so there are no consecutive odd numbers whose sum is 75.

13. If the first number is n, then the second is $5n$ and

$$\begin{aligned} n + 5n &= 15 + 4n \\ 2n &= 15 \\ n &= \frac{15}{2} \end{aligned}$$

The two numbers are $\frac{15}{2}$ and $\frac{75}{2}$.

14. If t is the number of hours each train traveled, then $40t$ is the number of miles traveled by the train going south, and $60t$ is the number of miles traveled by the train going north.

$$40t + 60t = 125$$

$$100t = 125$$

$$t = \frac{125}{100} = \frac{5}{4}$$

The time required was 1 hour and 15 minutes.

Answers to Review Problem Set; pages 432-447:

It is not anticipated that all of the exercises in the following review list will be used by any one teacher. Many teachers may choose to use some of them as supplementary or as "extra credit" exercises at the time the topic is studied earlier in the course.

In some cases it may be desirable to use portions of the list as a review list because the class is completing Part 2. In a few instances the completion of Chapter 10 may conclude the year's course. On the other hand, a teacher who plans to use Part 3 may not feel that review is necessary at this point and may omit the entire list.

In no instance is it recommended that an assignment include more than 3 or 4 difficult verbal problems.

1. (a) $\frac{4}{3}$
   (b) $\frac{10}{3}$
   (c) $-\frac{10}{3}$
   (d) $\frac{100}{33}$
   (e) 1
   (f) -1
   (g) $\frac{1}{\sqrt{2}}$
   (h) $\frac{1}{a^2 + 1}$
   (i) $x^2 + 4$
   (j) $\frac{1}{y^2 + 1}$
   (k) -2
   (l) $\frac{6}{5}$
   (m) 100
   (n) $-\frac{10}{68}$ or $-\frac{5}{34}$
   (o) $\frac{100}{45}$ or $\frac{20}{9}$

2. (a) 1
   (b) -1
   (c) $-1, 1$
   (d) $-1, 0$
   (e) $-1, 0$ (The expression is not a number if $a = -1$.)
   (f) has a reciprocal for every real number a
   (g) has a reciprocal for every real number a
   (h) -1

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(i) $-3, 3$

*3. $\big((a - 3)(a + 1)\big)\left(\frac{1}{a - 3}\right) = (a - 3)\left(\frac{1}{a - 3}\right)$

multiplication property of equality

$(a + 1)\left((a - 3)\left(\frac{1}{a - 3}\right)\right) = (a - 3)\left(\frac{1}{a - 3}\right)$

associative and commutative properties of multiplication

$(a + 1)\cdot 1 = 1$ definition of reciprocal

$a + 1 = 1$ multiplication property of 1

If $a = 3$, then $3 + 1 = 1$, and this is false.

We should not expect the sentence $a + 1 = 1$ to have the same truth set as the original sentence since our "multiplier" $\frac{1}{a - 3}$ is not a number when $a = 3$, and we used the multiplication property of equality in the very first step. In manipulating algebraic expressions, as in this example, we have to be constantly on guard that we do not become so engrossed in "pushing symbols" that we forget our algebraic structure. So long as we remember that $\frac{1}{a - 3}$ here is supposed to represent a number, we are safe in using algebraic properties. When we view $\frac{1}{a - 3}$ as a symbol only and apply our algebraic properties, any results we get can be only symbolic; to be interpreted as results about numbers, we have to check to see that we were actually using (symbolic) numbers at each step along the way.

4. (a) $-x^2 + 15x - 14$

(b) $a - 25$

(c) $8a^2 - 5a + 10$

(d) $10n + 13p - 13a$

(e) $7x^2 - 3x - 16$

(f) $-2b^2 + 2ab$

(g) $3a + 18b - 3c$

(h) $3x^2 - 8x + 19$

(i) $-12s + 4t - 10u$

5. Yes, in all cases

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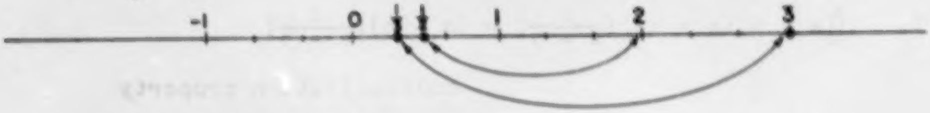
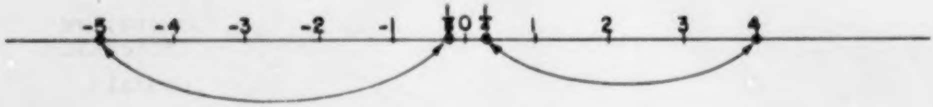
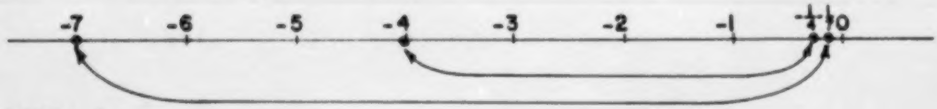
6. (a) $\frac{1}{3} < \frac{1}{2}$ is true.



(b) $-\frac{1}{5} < \frac{1}{4}$ is true.



(c) $-\frac{1}{7} < -\frac{1}{4}$ is false.



7. Yes, it is true. Let $a = 5$, $b = 2$. $(b < a)$

Then

$$\frac{1}{5} < \frac{1}{2} \qquad \left(\frac{1}{a} < \frac{1}{b}\right)$$

8. Yes, it is true. An example:

if $a = -4$, $b = -8$, $(b < a)$

then $\frac{1}{-8} > \frac{1}{-4}$. $\left(\frac{1}{a} < \frac{1}{b}\right)$

9. If a is positive and b is negative, then $\frac{1}{a} > \frac{1}{b}$, for the reciprocal of a positive number is a positive number and the reciprocal of a negative number is a negative number.

10. If $b < a$, then $a - b$ is positive. The proof of this follows.

If $b < a$, then

$b + (-b) < a + (-b)$	addition property of order
$b + (-b) < a - b$	definition of subtraction
$0 < a - b$	addition property of opposites

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If a is to the right of b on the number line, then the difference $a - b$ is positive.

11. If $(a - b)$ is a positive number, then $a > b$.
    If $(a - b)$ is a negative number, then $a < b$.
    If $(a - b)$ is zero then $a = b$.

12. If a, b, and c are real numbers, and $b < a$, then $b - c < a - c$. The proof of this follows:

    If $b < a$,

    $b + (-c) < a + (-c)$ addition property of order
    $b - c < a - c$ definition of subtraction

13. (a) $4a^2$
    (b) 2π
    (c) $19k$
    (d) $-3\sqrt{2}$
    (e) $4x$
    (f) $5x^2$
    (g) $-5a$
    (h) $-25pq - pq = (-25)pq + (-1)pq$
    $= -26pq$
    (i) $-16y$
    (j) 0
    (k) $-4y$
    (l) $3m$

14. $4 - 15 = -11$
    The resulting temperature is 11° below zero.

15. $(-50) - 30 = -80$
    The new position is 80 feet below the surface.

16. If the number is n, then

$$-16 = n - 25$$
$$-16 + 25 = n - 25 + 25$$
$$9 = n.$$

Hence, the number is 9.

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17. At 11 o'clock P.M., $t = -1$

At 2 o'clock A.M., $t = 2$.

$$2 - (-1) = 2 + 1 = 3$$

The interval is 3 hours.

At 6 o'clock A.M., $t = 6$.

At 4 o'clock A.M., the next day, $t = 28$.

$$28 - 6 = 22$$

The interval is 22 hours.

18. Let the distance in miles to the east of the 0 mark correspond to positive numbers:

	John's position on the number line	Rudy's position on the number line	The difference	Distance between them in miles
(a)	$(3)(10) = 30$	$(3)(-12) = -36$	$\lvert 30 - (-36) \rvert = 66$	66
(b)	$5 + 3(10) =$ $5 + 30 = 35$	$-6 + 3(12) =$ $-6 + 36 = 30$	$\lvert 35 - 30 \rvert = 5$	5
(c)	$(3)(10) = 30$	$(2\frac{3}{4})(-12) =$ $(\frac{11}{4})(-12) = -33$	$\lvert 30 - (-33) \rvert = 63$	63
(d)	$(3)(-10) = -30$	$-6 + 3(-12) = -42$	$\lvert (-30) - (-42) \rvert$ $= 12$	12

19. (a) If a is larger than 1, $0 < b < 1$.

- (b) If $0 < a < 1$, then $1 < b$.
- (c) If $a = 1$, then $b = 1$.
- (d) If $a = -1$, then $b = -1$.
- (e) If $a < -1$, then $-1 < b < 0$.
- (f) If $-1 < a < 0$, then $b < -1$.
- (g) If $a > 0$, then $b > 0$.
- (h) If $a < 0$, then $b < 0$.
- (i) Zero has no multiplicative inverse.
- (j) If b is the reciprocal of a, then a is the reciprocal of b.

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20. (a) 0

(b) No

(c) $n = 0$

(d) p can be any number including 0.

(e) Either $p = 0$ or $q = 0$, or both are zero.

(f) q must be 0.

(g) $x - 5$ must be zero since 7 is not zero.

(h) The given sentence is equivalent to

$$(9 \times 17 \times 3)y = 0$$

If $y > 0$, then the product $(9 \times 17 \times 3)y > 0$.
If $y < 0$, then the product $(9 \times 17 \times 3)y < 0$.

Therefore the only truth number of the given sentence is 0.

(i) $x - 8$ is zero when $x = 8$. It follows that 8 is a truth number of the sentence $(x - 8)(x - 3) = 0$.

$$(8 - 8)(8 - 3) = 0$$

$x - 3$ is zero when $x = 3$.
3 is a truth number of $(x - 8)(x - 3) = 0$.

The truth set of $(x - 8)(x - 3) = 0$ is $\{8, 3\}$.

21. (a) If $x = 20$, $x - 20$ is zero, and hence

$$(x - 20)(x - 100) \text{ is zero.}$$

If $x = 100$, $x - 100$ is zero, and hence,

$$(x - 20)(x - 100) \text{ is zero.}$$

The truth set is $\{20, 100\}$.

(b) $\{-6, -9\}$

(c) $\{0, 4\}$

(d) $\{-34\}$

(e) $\{1, 2, 3\}$

(f) $\{\frac{1}{2}, -\frac{3}{4}\}$

(g) $\{\frac{5}{3}, -\frac{1}{2}\}$

(h) $\{6\}$

(i) $\{-2\}$

(j) all real numbers greater than $-\frac{7}{2}$

(k) all real numbers less than $\frac{4}{3}$

(l) $\{\frac{15}{2}\}$

22. (a) 1

(b) $\frac{48}{13}$

(c) 10

(d) $-\frac{24a}{85}$

*(e) $\dfrac{\frac{3}{a} - \frac{5}{2a}}{\frac{3}{b} + \frac{5}{2b}} \cdot \dfrac{2ab}{2ab} = \dfrac{6b - 5b}{6a + 5a} = \dfrac{b}{11a}$

(f) -2

*(g) $\frac{19}{18}$

*(h) $\dfrac{\frac{x + 8}{9}}{\frac{3}{x + 2}} \cdot \dfrac{9(x + 2)}{9(x + 2)} = \dfrac{(x + 8)(x + 2)}{27}$

(i) $\dfrac{3y^2}{(y - 1)(y + 1)}$

(j) $\frac{19}{16}$

(k) $-\frac{7}{18}$

*(l) $\dfrac{a^2 + 8a + 12}{a^2 - 8a + 12}$ This may also properly be left in its original factored form.

(m) 1

(n) $\dfrac{1 - \frac{2}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{x^2 - 2x + 1}{x^2 - 1} = \dfrac{(x - 1)(x - 1)}{(x - 1)(x + 1)}$

$= \dfrac{x - 1}{x + 1}$

(o) $\dfrac{a^2 + b^2}{a + b}$

(p) $\dfrac{(x - 2)(x - 3)}{(-1)(x - 3)(-1)(x - 2)} = 1$

(q) 1

(r) $-\dfrac{x^2 + 3x + 2}{x^2 + 5x - 6}$

(s) $$\frac{\frac{2}{a - 1} + \frac{1}{a + 1}}{\frac{1}{a - 1} + \frac{2}{a + 1}} \cdot \frac{(a + 1)(a - 1)}{(a + 1)(a - 1)} = \frac{2(a + 1) + (a - 1)}{a + 1 + 2(a - 1)}$$

$$= \frac{3a + 1}{3a - 1}$$

*23. (a) $\{\frac{5}{2}\}$

(b) $\{\frac{13}{48}\}$

(c) $\{0\}$

(d) $$\frac{\frac{6x}{5} - \frac{x}{4}}{\frac{1}{2} + \frac{4}{5}} = x$$

$$\frac{\frac{6x}{5} - \frac{x}{4}}{\frac{1}{2} + \frac{4}{5}} \cdot \frac{20}{20} = x$$

$$\frac{24x - 5x}{10 + 16} = x$$

$$\frac{19x}{26} = x$$

$$\frac{19}{26}x - x = 0$$

$$x(\frac{19}{26} - 1) = 0$$

The truth set is $\{0\}$.

24. $(3x + 5)(2x - 3) = (3x + 5)(2x) + (3x + 5)(-3)$

Also $3x(2x - 3) + 5(2x - 3)$

25. $\left(25 + (-10)\right) + (-25)$

$= 25 + \left((-10) + (-25)\right)$ by the associative property of addition

$= 25 + \left((-25) + (-10)\right)$ by the commutative property of addition

$= \left(25 + (-25)\right) + (-10)$ by the associative property of addition

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26. $\frac{3}{8} \cdot \frac{5}{5} = \frac{15}{40}$ and $\frac{9}{20} \cdot \frac{2}{2} = \frac{18}{40}$

$\frac{15}{40} < \frac{18}{40}$; thus "$\frac{3}{8} < \frac{9}{20}$" is true.

$\frac{9}{20} \cdot \frac{3}{3} = \frac{27}{60}$ and $\frac{7}{15} \cdot \frac{4}{4} = \frac{28}{60}$

$\frac{27}{60} < \frac{28}{60}$; thus "$\frac{9}{20} < \frac{7}{15}$" is true.

Then, by the transitive property, $\frac{3}{8} < \frac{7}{15}$ is true.

27. Let e be the number of units in the length of each edge. Then $4e$ is the number of units in the perimeter and e^2 is the number of units in the area.

Now make the length of the edge $2e$.

New perimeter $8e$ The perimeter is multiplied by 2.
New area $4e^2$ The area is multiplied by 4.

28. (a) Set A is closed under multiplication.
Set B is closed under multiplication.

(b) $C = \{0, 1\}$. Set C is a subset of both set A and set B, but is a proper subset of B only.

29. $\frac{3x + 5}{2x - 7}$. The only value of x for which $\frac{3x + 5}{2x - 7}$ is not a real number is $x = \frac{7}{2}$. $2x - 7 = 0$ if and only if $x = \frac{7}{2}$.

The set of real numbers other than 0 is closed under division.

30. Since $\frac{a}{b} = \frac{2}{3}$, $a = \frac{2}{3}b$.

(a) If $a < 24$, then $\frac{2}{3}b < 24$
and $b < \frac{3}{2}(24)$

Hence b satisfies the inequality $b < 36$.

(b) $a < 16$

31. If a is one of the numbers, then $a \neq 0$ and $\frac{2}{a}$ is the other number.

(a) $a < 3$. If $0 < a < 3$, then $\frac{1}{a} > \frac{1}{3}$ and $\frac{2}{a} > \frac{2}{3}$, by the multiplication property of order. If $a < 0$, then $\frac{2}{a} < 0$. Thus, the other number is greater than $\frac{2}{3}$ or less than 0.

(b) $a < -3$. Here $\frac{1}{a} > -\frac{1}{3}$ and $\frac{2}{a} > -\frac{2}{3}$. Also, since $a < 0$, $\frac{2}{a} < 0$. Thus, the other number is greater than $-\frac{2}{3}$ and less than 0.

32. No

$$(a \div b) \div c = (a \cdot \tfrac{1}{b}) \cdot \tfrac{1}{c}$$

$$= \frac{a}{bc}$$

$$a \div (b \div c) = a \div (b \cdot \tfrac{1}{c})$$

$$= a \cdot \frac{1}{b \cdot \frac{1}{c}}$$

$$= \frac{ac}{b}$$

$\frac{a}{bc}$ and $\frac{ac}{b}$ are not equal for all values of a, b, and c. (For example, let $a = b = c = 2$)

33. No A counter example: $2 \div 3 \neq 3 \div 2$, since $\frac{2}{3} \neq \frac{3}{2}$

34. If $x = a + \frac{1}{a}$ and $a = \frac{1}{2}$, then

$$ax + a^2 = \tfrac{1}{2}\left(\tfrac{1}{2} + \frac{1}{\frac{1}{2}}\right) + \left(\tfrac{1}{2}\right)^2$$

$$= \tfrac{1}{4} + 1 + \tfrac{1}{4}$$

$$= 1\tfrac{1}{2} \text{ or } \tfrac{3}{2}$$

35. If a is between p and q, then $\frac{1}{a}$ is between $\frac{1}{p}$ and $\frac{1}{q}$. Suppose $p > q$, then $q < a < p$. Since $a < p$, $\frac{1}{a} > \frac{1}{p}$. Since $a > q$, $\frac{1}{a} < \frac{1}{q}$. Hence, $\frac{1}{a}$ is between $\frac{1}{p}$ and $\frac{1}{q}$.

36. (a) $18y$ feet, if $6y$ is the number of yards
(b) $24f$ inches, if $2f$ is the number of feet
(c) $8k$ pints, if $4k$ is the number of quarts
(d) $(n - 10)$ years, if she is now n years old
(e) $(16k + t)$ ounces, if k is the number of pounds and t is the number of ounces
(f) $144f$ square inches, if f is the number of square feet
(g) $(100d + 25k)$ cents, if d is the number of dollars and k is the number of quarters
(h) $(100d + 25k + 10t + 5n)$ cents, if there are d dollars, k quarters, t dimes, and n nickels
(i) $n + 1$, if n is the whole number
(j) $\frac{1}{n}$, if the number is n
(k) $5280k$ feet, if k is the number of miles
(l) $2(5280k)$ feet, if k is the number of miles

37. In these open sentences, the phrases and numbers often give a clue to the possible translations. In each part, just one interpretation is given, for suggestive purposes only, and there is no implication that this interpretation is the "best" one; pupils should be encouraged to look for more than one meaningful translation. Note that with certain translations the variable is restricted to the set of whole numbers, whereas with other translations there is no such restriction.

(a) My grandfather is less than 80 years old.

(b) His annual salary is 3600 dollars.

(c) The assets of a certain bank are more than one hundred million dollars.

(d) The sum of the angles of a triangle is 180°.

(e) The length of a rectangle is 18 inches more than the width. The area is 360 square inches.

(f) The length of a rectangle is three times the width and the area does not exceed 300 square inches.

(g) The number of units in the length of a rectangle is two more than the number of units in the width. A side of a square is one unit longer than the width of the rectangle. The area of the square is greater than the area of the rectangle.

(h) Farmer Jones had 30 sheep which he expected to sell for $20.00 a head; some of the sheep died, but he sold the remainder for $24 a head, receiving as much as or more than he had originally expected.

(i) The sides of an equilateral triangle and a square are such that the perimeter of the triangle is equal to the perimeter of the square.

(j) The sum of five consecutive numbers is less than 90, and the least of the numbers is greater than 13.

In each case above the response could have been given in the form of one sentence by use of connectives. Sometimes, for the sake of clarity, it is better to use several shorter sentences in making a translation.

38. (a) If n is the number, then the number diminished by 3 is $n - 3$.

(b) If t is the first temperature, the temperature after it rises 20 degrees is $t + 20$ degrees.

(c) If n is the number of pencils purchased at 5 cents each, the cost is $5n$ cents.

(d) If the number of nickels in my pocket is y and the number of dimes is x, the amount of money I have is $(10x + 5y + 6)$ cents.

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(e) If the number is n, then the result of increasing it by twice the number is $n + 2n$.

(f) If the first number is x and the other is y, the first increased by twice the second is $x + 2y$.

(g) If the number of weeks is w, the number of days is $7w$.

(h) If x is the number of melons and y is the number of pounds of hamburger, the total cost is $29x + 59y$ cents.

(i) If n is the number of inches in the shorter side of a rectangle, $n + 3$ is the number of inches in the longer side, and the area is $n(n + 3)$ square inches.

(j) If x is the population of the city in Kansas, then one million more than twice the population is $2x + 1{,}000{,}000$.

(k) If x is the number of dollars salary per month, the annual salary is $12x$ dollars.

(l) If b is the number of dollars in Betty's allowance, the number of dollars in Arthur's allowance is $2b + 1$.

(m) If h is the number of hours, the distance traveled at 40 m.p.h. is $40h$ miles.

(n) If the number of dollars in the value of the property is y, the real estate tax is $\frac{y}{1000}(25)$ dollars.

(o) If the number of pounds Earl weighs is e, the number of pounds Donald weighs is $e + 40$.

(p) $r - 1$ is the number of miles the first car travels in an hour, if r is the number of miles the following car travels in an hour.

(q) If x is the number of pounds of steak, the cost in dollars is $1.59x$.

(r) If the number of hours Catherine works is z, the number of dollars she earns is $.75z$.

(s) If the number of gallons is g, the cost in cents is $33.2g$.

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39. (a) y is Mary's sister's age.
$16 = y + 4$

(b) b is the number of bananas.
$9b = 54$

(c) n is the number.
$2n + n < 39$

(d) b is the number of dollars in Betty's allowance.
$2b + 1$ is the number of dollars in Arthur's allowance.
$2b + 1 = 3b - 2$

(e) t is the number of hours.
$40t = 260$

(f) t is the number of hours the trip took.
$50t > 300$, if we assume that the maximum speed is not maintained for the entire trip, or
$50t = 300$, if we assume that the maximum speed is maintained. The sentence $50t \geq 300$ gives a correct translation.

(g) h is the number of feet of elevation of Pike's Peak.
$h > 14{,}000$

(h) n is the number of pages in the book.
$1.4 = 0.003n + 2(.1)$

(i) Let p be the number of people in any city in Colorado.
$3{,}000{,}000 > 2p + 1{,}000{,}000$

(j) $x^2 < (x - 1)(x + 1)$. This is a correct translation. However, it is not possible to find any value of x for which it is true.
Using the distributive property we get:
$x^2 < x^2 - x + x - 1$,
$x^2 < x^2 - 1$, and this is false for every x.

(k) y is the number of dollars in the valuation of the property.

$\frac{y}{1000}(24.00) = 348.00$, or $.024y = 348.00$

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(l) w is the number of pounds Earl weighs.

$152 \geq w + 40$

(m) n is the counting number.

n + 1 is its successor.

$n + (n + 1) = 575$

(n) n is the counting number.

n + 1 is its successor.

$n + (n + 1) = 576$. This sentence is false for all counting numbers. If a number is odd, its successor is even; if the number is even, its successor is odd; in either case, their sum cannot be even.

(o) n is the first number.

n + 1 is the second number.

$n + (n + 1) = 576$.

Here the solution set is not the empty set since the domain of n is not restricted to the counting numbers.

(p) f is the number of feet in the length of one piece of board.

2f + 1 is the number of feet in the length of the other piece.

$f + (2f + 1) = 16$

(q) $3x = 225$.

(r) $C + 1 = T$

(s) y is the number of years old Mary is now.

y - 6 is the number of years old Mary was six years ago.

y + 4 is the number of years old Mary will be in four years.

$y + 4 = 2(y - 6)$

(t) t is the ten's digit.

u is the unit's digit.

10t + u is the number.

u + t is the sum of the digits.

$10t + u = 3(u + t) + 7$

(u) n is the number.
$3(n + 17) = 192$

(v) $3(n + 17) < 192$

40. m is the number of months that have elapsed since his weight was 200 lbs.
$175 + 5m = 200$

41. (a) n is the number.
$n \leq 7$ and $n \geq 1$

(b) b is the number Betty chooses, and $b \leq 7$.
n is the number Paul chooses, and $n \leq 5$.
Both are counting numbers, so $b > 0$ and $n > 0$.
If $b = 1$ and $n = 1$, $b + 3n = 4$; if $b = 7$ and $n = 5$, $b + 3n = 22$; hence:
$b + 3n \geq 4$ and $b + 3n \leq 22$.

(c) b is the number Betty chooses, and $b \leq 7$.
n is the number Paul chooses, and $n \leq 5$.
Betty chooses a counting number, so $b > 0$.
Paul chooses a whole number, so $n \geq 0$.
Hence: $b + 3n \geq 1$ and $b + 3n \leq 22$.

42. (a) The fee for 4 hours is 35¢ + 3(20)¢ or 95¢.

(b) t is the number of one-hour periods after the initial hour.
$35 + 20t$ is the parking fee.

(c) h is the total number of one-hour periods parked.
$h - 1$ is the number of one-hour periods after the initial hour.
$35 + 20(h - 1)$ is the parking fee.

43. (a) $100x + 40y$ is the total number of gallons.

(b) $120(100)$ is the number of gallons from the first pipe in 2 hours.
$40y$ is the number of gallons from the second pipe in y minutes, where $y > 120$.

$120(100) + 40y$ is the total number of gallons in y minutes, $y > 120$.

*(c) $100x + 40y = 20{,}000$

If x is 0, 60, 120, 160, 180, 200

and y is 500, 350, 200, 100, 50, 0

the sentence is true.

44. c is the number of degrees Centigrade.

$1.8c + 32$ is the number of degrees Fahrenheit.

$1.8c + 32 < 50$

$c < \frac{50 - 32}{1.8}$

45. d is the number of dollars Harry receives.

$d + 15$ is the number of dollars Dick receives.

$2(d + 15)$ is the number of dollars Tom receives.

$$\begin{aligned} d + (d + 15) + 2(d + 15) &= 205 \\ 4d + 45 &= 205 \\ 4d &= 160 \\ d &= 40 \end{aligned}$$

Harry must receive $40.

Dick must receive $55.

Tom must receive $110.

46. Last year's cost was $100d$ cents per dozen.

This year's cost is $100d + c$ cents per dozen.

Half a dozen balls will cost $\frac{100d + c}{2}$ cents.

47. Since the amounts are proportional to the ages 7 and 3, they may be represented as $7x$ dollars and $3x$ dollars.

$$\begin{aligned} 7x + 3x &= 24 \\ 10x &= 24 \\ x &= 2.40 \end{aligned}$$

Then $7x = 16.80$ and $3x = 7.20$.

The older child receives $16.80 and the younger, $7.20.

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48. Let x be the new average.
Then $8x$ is the total number of points received by the 8 pupils who remained in the class.
The total number of points received by the 10 pupils is 720. Hence,

$$\begin{aligned} 8x + 192 &= 720, \\ 8x &= 528, \\ \text{and} \qquad x &= 66. \end{aligned}$$

Hence the new average is 66.

49. $S = [\ldots, -4, -2, 0, 2, 4, \ldots]$

Addition, subtraction or multiplication of any two numbers of the set gives a number of the set.

Division may not give a number of the set. For example, $\frac{2}{6}$ is not an even integer.

Finding the average of pairs of numbers from the set may not give a number of the set. For example, $\frac{2 + 4}{2}$ or 3 is not an even integer. Thus, the set of even integers is closed under addition, subtraction and multiplication, but is not closed under division or pairwise averaging.

50. If the first shirt cost x dollars, then

$$\begin{aligned} x - .25x &= 3.75, \\ .75x &= 3.75, \\ x &= 5. \end{aligned}$$

The first shirt cost $5, so he lost $1.25 on it.

If the second shirt cost y dollars, then

$$\begin{aligned} y + .25y &= 3.75, \\ 1.25y &= 3.75, \\ y &= 3. \end{aligned}$$

The second shirt cost $3, so he gained $0.75 on it.

Thus, he lost $0.50 on the sale of the two shirts.

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51. If n is the number of nickels,
then $12 - n$ is the number of dimes,
$5n$ is the number of cents in n nickels,
and $10(12 - n)$ is the number of cents in $(12 - n)$ dimes.
Since the total number of cents is 95, we have

$$\begin{aligned} 5n + 10(12 - n) &= 95 \\ 5n + 120 - 10n &= 95 \\ -5n &= -25 \\ -\frac{1}{5}(-5n) &= -\frac{1}{5}(-25) \\ n &= 5 \end{aligned}$$

There were 5 nickels and 7 dimes.

52. If t is the number of hours he rides into the woods,
then $5 - t$ is the number of hours to ride out.
$4t$ is the number of miles he went one way
and so is $15(5 - t)$. Hence,

$$\begin{aligned} 4t &= 15(5 - t) \\ 4t &= 75 - 15t \\ 19t &= 75 \\ t &= \frac{75}{19} \text{ or } 3\frac{18}{19} \end{aligned}$$

He can ride in for $3\frac{18}{19}$ hours so he can go a distance of $(3\frac{18}{19} \times 4)$ miles into the woods.

53. If s is the speed of the wind in miles per hour,
then the speed of the plane is $200 - s$ miles per hour,
and the distance traveled is $3\frac{1}{2}(200 - s)$ miles.
So,

$$\begin{aligned} 3\frac{1}{2}(200 - s) &= 630 \\ 700 - 3\frac{1}{2}s &= 630 \\ -3\frac{1}{2}s &= -70 \\ s &= -70(-\frac{2}{7}) \\ s &= 20 \end{aligned}$$

The speed of the wind is 20 miles per hour.

Suggested Test Items

1. Simplify each of the following:

(a) $-4 - 6$

(b) $12 - 4 - 3$

(c) $\frac{21}{-7}$

(d) $\frac{5}{27} - \frac{1}{9}$

(e) $12 - 4 \cdot \frac{1}{5} + \frac{4}{5}$

(f) $\frac{3x}{4} \cdot \frac{-4}{3x}$, where $x \neq 0$

(g) $\frac{2(x + 2)}{x - 2} \cdot \frac{1}{4(x + 2)}$, where $x \neq 2$ and $x \neq -2$

(h) $\frac{3y}{\frac{1}{2}y}$, where $y \neq 0$

(i) $(2a - 1) - (a + 2)$

(j) $3(x - 2) - (x - 2)$

(k) $\frac{2z + 2}{z + 1}$, where $z \neq -1$

(l) $\frac{\frac{x - 5}{3}}{\frac{x - 5}{6}}$, where $x \neq 5$

(m) $3y - 7x + 5 + 2x - y - 2$

2. If $m = -4$ and $n = 3$ find the value of

(a) $m - n$

(b) $n - m$

(c) $|m - n|$

(d) $|n - m|$

(e) $\frac{m - n}{m + n}$

(f) $(m - n)(\frac{1}{m + n})$

3. Simplify each of the following:

(a) $\frac{3a^2b}{c} \cdot \frac{5c^2}{6ab}$

(b) $\frac{a - b}{2b - 2a}$

(c) $\frac{x + 1}{3} + \frac{x - 1}{4}$

(d) $\frac{b}{b - \frac{4}{b}}$

4. For what values of the variables in Problem 3 is each of these expressions not a real number?

5. Find the truth set of each of the following:

(a) $\frac{x}{3} + \frac{x}{4} = 7$

(b) $\frac{x}{6} = \frac{x + 2}{5}$

(c) $9 - \frac{1}{4}x = 2x$

(d) $\frac{1}{y} = \frac{2}{5}$

(e) $\frac{x}{3} - \frac{2}{5} = \frac{1}{2}$

6. For what values of the variable is each of the following true?

(a) $0 \cdot x = 0$

(b) $5x = 0$

(c) $x \cdot 0 = 3$

(d) $|x - 2| = 0$

(e) $|x - 2| > 0$

7. Find the truth set of each of the following:

(a) $y - 3 = 3 - y$

(b) $1 + x > 1 - x$

(c) $\frac{2}{|x|} > 1$

(d) $|x| > 2|x|$

8. If $a < b$, which of the following numbers are positive?

(a) $a - b$

(b) $\frac{1}{a - b}$

(c) $|a - b|$

(d) a, if $ab < 0$

(e) $(b - a)a^2$

9. What number must be added to $-2x + 3y - 4$ to get $x - 2y + 2$?

10. By what number must $\frac{2}{a}$ be multiplied to get $3ab$?

11. If the numerator and the denominator of the fraction $\frac{3}{4}$ are each increased by x, where x is positive, the value of the fraction is increased by $\frac{1}{12}$. Find x.

12. A student lived at a boarding house, where he paid rent at the rate of \$1.50 per day, except on those days when he was able to work for the boarding house owner. Whenever he worked for the owner for a day, the owner charged him no rent for that day, and gave him \$8 credit toward his rent for the month. The student paid \$8.50 rent for the month of January. Write and solve an open sentence to find out how many days he worked for the owner that month. (Hint: If the student worked n days, for how many days did he pay rent?)

13. Horatio is making a scale model of a building. If the scale is $\frac{1}{60}$ --that is, if a length of 60 feet on the building is represented by a length of one foot on the model--how long should he make the wall of his model which is to correspond to a 25-foot wall of the building? Write and solve an open sentence for this problem.

Answers to Suggested Test Items

1. (a) -10
   (b) 5
   (c) -3
   (d) $\frac{2}{27}$
   (e) 12
   (f) -1
   (g) $\frac{1}{2(x-2)}$ or $\frac{1}{2x-4}$
   (h) 6
   (i) $a - 3$
   (j) $2x - 4$
   (k) 2
   (l) 2
   (m) $2y - 5x + 3$

2. (a) -7
   (b) 7
   (c) 7
   (d) 7
   (e) 7
   (f) 7

3. (a) $\frac{5ac}{2}$
   (b) $-\frac{1}{2}$
   (c) $\frac{7x+1}{12}$
   (d) $\frac{b^2}{b^2-4}$

4. (a) If at least one of a, b, and c is 0
   (b) If $a = b$
   (c) None
   (d) If b is -2, 0, or 2

5. (a) $\{12\}$
   (b) $\{-12\}$
   (c) $\{4\}$
   (d) $\{\frac{5}{2}\}$
   (e) $\{\frac{27}{10}\}$

6. (a) all real values
   (b) 0
   (c) for no value of x
   (d) 2
   (e) all real values except 2

7. (a) $\{3\}$

(b) all positive real numbers

(c) all values of x which are greater than -2 and less than 2, except 0.

(d) $\emptyset$

8. The numbers in (c) and (e) are positive; the others negative.

9. $x - 2y + 2 - (-2x + 3y - 4) = 3x - 5y + 6$

10. $$\frac{3ab}{\frac{2}{a}} = \frac{3a^2b}{2}$$

11. $$\frac{3 + x}{4 + x} - \frac{3}{4} = \frac{1}{12}$$

$$\begin{aligned} 12(3 + x) - 3\cdot 3(4 + x) &= 4 + x \\ x &= 2 \end{aligned}$$

12. If the student worked n days, then he paid rent for $(31 - n)$ days. Then

$$\begin{aligned} 1.50(31 - n) - 8(n) &= 8.50 \\ 46.50 - 1.50n - 8n &= 8.50 \\ 46.50 - 9.5n &= 8.50 \\ 38 &= 9.5n \\ 4 &= n \\ &\{4\} \end{aligned}$$

The student worked four days during the month of January.

13. If the wall of the model is y feet long, then

$$\begin{aligned} \frac{1}{60} &= \frac{y}{25} \\ 300\left(\frac{1}{60}\right) &= 300\left(\frac{y}{25}\right) \\ 5 &= 12y \\ \frac{5}{12} &= y \\ &\left\{\frac{5}{12}\right\} \end{aligned}$$

The wall must be $\frac{5}{12}$ ft. (5 inches) long.

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Challenge Problems

Part 2

1. The relation "$\succ$" does not have the comparison property. For example, 2 and -2 are different real numbers, but neither is further from 0 than the other; in other words, neither of the statements "$-2 \succ 2$" and "$2 \succ -2$" is true.

   The transitive property for "$\succ$" would read: If a, b, and c are real numbers for which $a \succ b$ and $b \succ c$, then $a \succ c$. This is certainly a true statement as can be seen by substituting the phrase "is further from 0 than" for "$\succ$" wherever it occurs.

   The relations "$\succ$" and "$>$" have the same meaning for the numbers of arithmetic: "is further from 0 than" and "is to the right of" mean the same thing on the arithmetic number line.

2. By the definition of the product of two real numbers, we have

$$ab = |a| \cdot |b| \quad \text{or} \quad ab = -(|a| \cdot |b|).$$

(i) If $ab = |a| \cdot |b|$, then

$$\begin{aligned} |ab| &= \big|\,|a| \cdot |b|\,\big| \\ &= |a| \cdot |b|, \qquad \text{since } |a|\cdot|b| \geq 0. \end{aligned}$$

(ii) If $ab = -(|a| \cdot |b|)$, then

$$\begin{aligned} |ab| &= |-(|a| \cdot |b|)| \\ &= \big|\,|a| \cdot |b|\,\big|, \qquad \text{since } |x| = |-x| \text{ for all } x \\ &= |a| \cdot |b|. \end{aligned}$$

3. Prove that the number 0 has no reciprocal.
Proof: Assume that the sentence of the theorem is false. Then 0 has a reciprocal, say a. This would mean that

$$0 \times a = 1.$$

Since the product of zero and any real number is zero, it follows that

$$0 = 1.$$

This sentence is false. Thus our assumption that zero has a reciprocal is a false assumption, and it follows that zero has no reciprocal

4. Prove that the reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.
Proof: The statement follows immediately from the definition, $a \times \frac{1}{a} = 1$, since the product of two numbers is positive if and only if both numbers are positive or both numbers are negative. (Proof by contradiction would also be possible.)

5. Prove that the reciprocal of the reciprocal of a non-zero real number a is a.
Proof: Since $\frac{1}{\frac{1}{a}}$ is the reciprocal of $\frac{1}{a}$ by the definition of a reciprocal, it follows that $(\frac{1}{a})(\frac{1}{\frac{1}{a}}) = 1$.

Similarly, since $\frac{1}{a}$ is the reciprocal of a, it follows that $(a)(\frac{1}{a}) = 1$, or, by the commutative property, $(\frac{1}{a})(a) = 1$.

Compare $(\frac{1}{a})(\frac{1}{\frac{1}{a}}) = 1$ with $(\frac{1}{a})(a) = 1$. We see that the number $\frac{1}{a}$ has reciprocals $\frac{1}{\frac{1}{a}}$ and a. Since any non-zero real number has only one reciprocal, it follows that

$\frac{1}{\frac{1}{a}} = a$, which is what we wanted to prove.

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6. (a) $|9 - 2| = |9| - |2|$

(b) $|2 - 9| > |2| - |9|$

(c) $|9 - (-2)| > |9| - |-2|$

(d) $|(-2) - 9| > |-2| - |9|$

(e) $|(-9) - 2| > |-9| - |2|$

(f) $|2 - (-9)| > |2| - |-9|$

(g) $|(-9) - (-2)| = |-9| - |-2|$

(h) $|(-2) - |-9)| > |-2| - |-9|$

7. From the preceding exercise the student will, we hope, infer that for all real numbers a and b,

$$|a - b| \geq |a| - |b|$$
$$|a - b| \geq |b| - |a|$$
$$|a - b| \geq \big||a| - |b|\big|.$$

In case some of the more capable students are interested in seeing a proof of these statements, we give the following. The statement that $|x + y| \leq |x| + |y|$ for all real numbers x and y can be used to prove the three statements above: With $x = a - b$ and $y = b$, we have

$$|a| = |(a - b) + b| \leq |a - b| + |b|.$$

By the addition property of order.

$$|a| + (-|b|) \leq |a - b|$$
$$|a| - |b| \leq |a - b|$$
$$|a - b| \geq |a| - |b|.$$

Similarly, $x = b - a$ and $y = a$ leads to the sentence

$$|b - a| \geq |b| - |a|.$$

Since $|b - a| = |-(b - a)| = |a - b|$, this gives

$$|a - b| \geq |b| - |a|.$$

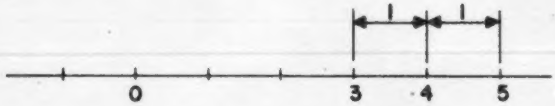
But $|b| - |a| = -(|a| - |b|)$, so that we now have

$$|a - b| \geq |a| - |b|,$$
$$|a - b| \geq -(|a| - |b|).$$

Therefore, $|a - b| \geq \big||a| - |b|\big|$.

8. The distance between a and b is found to be at least as great as the distance between $|a|$ and $|b|$, because a and b can be on opposite sides of 0, while $|a|$ and $|b|$ must be on the same side.

9. The two numbers are 3 and 5.

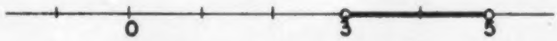


Though the above is the suggested approach to this problem, some students may try to do it by using the definition of absolute value.

If $|x - 4|$ is 1, that is, $|x - 4|$ is another name for 1, then $(x - 4)$ must, by definition of absolute value, be either 1 or -1. Thus,

$$x - 4 = 1 \quad \text{or} \quad x - 4 = -1$$
$$x = 5 \quad \text{or} \quad x = 3.$$

10. The truth set of the sentence $|x - 4| < 1$ is the set $3 < x < 5$.



Rather than using formal methods for solution of the inequality, the student will be guided by the question: What is the set of numbers x such that the distance between x and 4 is less than 1? As in the case of the preceding exercise, the student may work directly from the definition of absolute value instead of by the suggested approach.

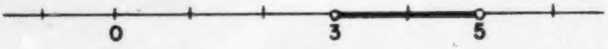
For example:

If $x - 4 \geq 0$, then $|x - 4| = x - 4$
But $|x - 4| < 1$
So $x < 5$

If $x - 4 < 0$, then $|x - 4| = -(x - 4)$
$= -x + 4$
But $|x - 4| < 1$
So $-x + 4 < 1$
$-x < -3$
$x > 3$

Thus, $x \geq 4$ and $x < 5$
or $x < 4$ and $x > 3$.
Finally, $3 < x < 5$.

11. The graph of the truth set of $x > 3$ and $x < 5$ is

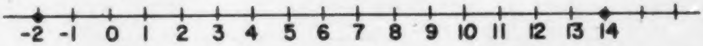


It is the same as the truth set of $|x - 4| < 1$.

12. In some of the following exercises the methods described in connection with the solution of Problems 6 and 7 above may be used by the students. The method of the distance on the number line is our main objective here.

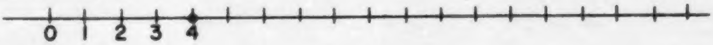
(a) Truth set: $\{-2, 14\}$

Graph:



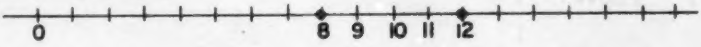
(b) Truth set: $\{4\}$

Graph:



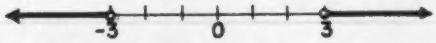
(c) Truth set: $\{8, 12\}$

Graph:



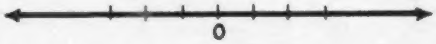
(d) Truth set: Real numbers x such that $x < -3$ or $x > 3$.

Graph:



(e) Truth set: All real numbers

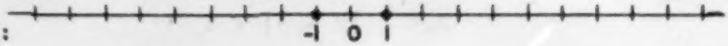
Graph:



(f) $|y| = 1$

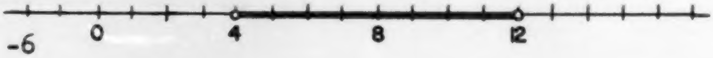
Truth set: $\{-1, 1\}$

Graph:



(g) Truth set: Real numbers y such that $4 < y < 12$.

Graph:

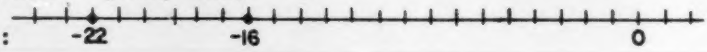


(h) $|z| = -6$

The empty set $\emptyset$

(i) Truth set: $\{-22, -16\}$

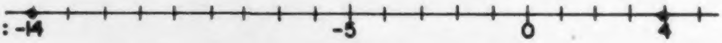
Graph:



(j) $|y + 5| = |y - (-5)| = 9$

Truth set: $\{-14, 4\}$

Graph:



13. Prove: $\frac{1}{(-a)} = -\left(\frac{1}{a}\right)$

Proof:

$\frac{1}{-a} = \frac{1}{(-1)a}$ $\quad$ $-a = (-1)a$

$= \frac{1}{-1} \cdot \frac{1}{a}$ $\quad$ $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$

$= (-1) \cdot \frac{1}{a}$ $\quad$ Definition of multiplicative inverse

$= -\left(\frac{1}{a}\right)$ $\quad$ $(-1)x = -x$

14. Prove: If $a < b$, a and b both positive real numbers, then $\frac{1}{b} < \frac{1}{a}$.

Proof:

$a < b$ $\quad$ Given

$a\left(\frac{1}{a} \cdot \frac{1}{b}\right) < b\left(\frac{1}{a} \cdot \frac{1}{b}\right)$ $\quad$ Multiplication property of order; $\frac{1}{a} \cdot \frac{1}{b}$ is positive, since a and b are positive.

$\left(a \cdot \frac{1}{a}\right)\frac{1}{b} < \left(b \cdot \frac{1}{b}\right)\frac{1}{a}$ $\quad$ Associative and commutative properties of multiplication

$1(\frac{1}{b}) < 1(\frac{1}{a})$	Definition of multiplicative inverse
$\frac{1}{b} < \frac{1}{a}$	Multiplication property of 1

15. Prove: If $a < b$, where a and b are both negative real numbers, then $\frac{1}{b} < \frac{1}{a}$.

Proof:	
$a < b$	Given
$a(\frac{1}{a} \cdot \frac{1}{b}) < b(\frac{1}{a} \cdot \frac{1}{b})$	Multiplication property of order

(Since $\frac{1}{a}$ and $\frac{1}{b}$ are both negative numbers, $(\frac{1}{a} \cdot \frac{1}{b})$ is a positive number.) The remainder of the proof is identical to that in Problem 14. Alternatively, since $a < b$, $-a > -b$. Because $-a$ and $-b$ are both positive numbers, Problem 14 allows us to assert that $\frac{1}{-a} < \frac{1}{-b}$, and $-\frac{1}{a} < -\frac{1}{b}$. Taking opposites, again we have,

$$\frac{1}{a} > \frac{1}{b}.$$

16. If $a < 0$ and $b > 0$, then $\frac{1}{a} < \frac{1}{b}$ because $\frac{1}{a}$ is negative and $\frac{1}{b}$ is positive.

17.

$\frac{a}{c} + \frac{b}{c} = a(\frac{1}{c}) + b(\frac{1}{c})$	Definition of division
$= (a + b)\frac{1}{c}$	Distributive property
$= \frac{a + b}{c}$	Definition of division

18.

$\frac{a}{c} + \frac{b}{d} = \frac{a}{c}(\frac{d}{d}) + \frac{b}{d}(\frac{c}{c})$	Multiplication property of 1
$= \frac{ad}{cd} + \frac{bc}{cd}$	Commutative property of multiplication and the theorem: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, $b \neq 0$, $d \neq 0$
$= \frac{ad + bc}{cd}$	Proved in Problem 17

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19. (a) Yes, because the product of any two numbers of the set is a member of the set.

(b) $(-1) \times j = -j$; $j \times (-1) = -j$

Hence, $(-1) \times j = j \times (-1)$

$j \times (-j) = 1$; $(-j) \times j = 1$

Hence, $j \times (-j) = (-j) \times j$

$(-1) \times (-j) = j$; $(-j) \times (-1) = j$

Hence, $(-1) \times (-j) = (-j) \times (-1)$

(c) $(-1) \times j \;\times (-j) = (-j) \times (-j) = -1$

$(-1) \times \; j \times (-j) \;= (-1) \times 1 = -1$

Hence, $(-1) \times j \;\times (-j) = (-1) \times \; j \times (-j)$

$1 \times \; (-1) \times j \;= (-1) \times j = -j$

$1 \times \; (-1) \times j \;= 1 \times (-j) = -j$

Hence, $1 \times (-1) \;\times j = 1 \times \; (-1) \times j$

(d) Yes.

$1 \times 1 = 1$

$(-1) \times 1 = -1$

$j \times 1 = j$

$(-j) \times 1 = -j$

(e) $1 \times 1 = 1$. Hence, 1 is the reciprocal of 1.

$(-1) \times (-1) = 1$. Hence, -1 is the reciprocal of -1.

$j \times (-j) = 1$. Hence, $-j$ is the reciprocal of j.

$(-j) \times j = 1$. Hence, j is the reciprocal of $-j$.

(f) If x is a number such that $j \times x = 1$, then

$$(-j) \times (j \times x) = (-j) \times 1$$
$$(-j) \times j \;\times x = (-j) \times 1$$
$$1 \times x = -j$$
$$x = -j$$

If $x = -j$, then $j \times x = j \times (-j) = 1$.
Hence, the truth set is $\{-j\}$.

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(g) Similarly, the truth set is $\{-1\}$. Multiply by j, since j is the reciprocal of $-j$.

(h) The truth set is $\{1\}$. Multiply by (-1), since (-1) is the reciprocal of j^2 or (-1).

(i) The truth set is $\{1\}$. $j^3 = (j^2) \times j = (-1) \times j = -j$. Hence, multiply by j, since j is the reciprocal of $-j$.

20. A rate of 3 minutes per mile is 20 m.p.h. Thus, the time going is $\frac{360}{20}$, or 18 hours. 3 miles per minute is 180 miles per hour. Thus the time returning is $\frac{360}{180}$, or 2 hours. The total time is $18 + 2$, or 20, hours, the total distance $2 \cdot 360$, or 720, miles and the average rate is $\frac{720}{20}$, or 36 m.p.h.

21. Start with the sum t.
Then, for the ten numbers, the new sum is

$$3(t + 10 \cdot 4) - 10 \cdot 4 = 3t + 80.$$

For 20 numbers,

$$3(t + 20 \cdot 4) - 20 \cdot 4 = 3t + 160.$$

The new sum, then, is 160 more than three times the original sum.